Time-Varying General Dynamic Factor Models and the Measurement of Financial Connectedness Technical Appendix

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A1 Notation

Denote by $\|\boldsymbol{A}\| = \sqrt{\mu^{(1)}(\boldsymbol{A}^\dagger \boldsymbol{A})}$, where $\mu^{(1)}(\boldsymbol{A}^\dagger \boldsymbol{A})$ is the largest eigenvalue (which is always real) of $\boldsymbol{A}^\dagger \boldsymbol{A}$, the spectral norm of a given complex $p \times p$ matrix $\boldsymbol{A}$ and by $\|\boldsymbol{A}\|_F = \sqrt{\mathrm{tr}(\boldsymbol{A}^\dagger \boldsymbol{A})}$ its Frobenius norm. Similarly, write $\|\boldsymbol{v}\| = \sqrt{\sum_{i=1}^p v_i^2}$ for the Euclidean norm of a p-dimensional vector $\boldsymbol{v} = (v_1, \ldots, v_p)'$

A2 Proof of Lemma 1

Let $\gamma_{ij}^{\chi}(\tau;\ell) = \mathrm{E}[\chi_{it}(\tau)\chi_{jt-\ell}(\tau)]$ and $\gamma_{ij}^{\xi}(\tau;\ell) = \mathrm{E}[\xi_{it}(\tau)\xi_{jt-\ell}(\tau)]$. By Assumptions (A1), (B1), and (B4), for any $\ell \in \mathbb{Z}$,

$$
\begin{aligned}
\left|\frac{\mathrm{d}^2\gamma_{ij}^{\chi}(\tau;\ell)}{\mathrm{d}\tau^2}\right| &= \left|\sum_{k=0}^{\infty}\sum_{s=1}^{q}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left\{c_{isk}(\tau)c_{js,k+|\ell|}(\tau)\right\}\right| \\
&= \left|\sum_{k=0}^{\infty}\sum_{s=1}^{q}\Big[\frac{\mathrm{d}^2c_{isk}(\tau)}{\mathrm{d}\tau^2}c_{js,k+|\ell|}(\tau) + \frac{\mathrm{d}^2c_{js,k+|\ell|}(\tau)}{\mathrm{d}\tau^2}c_{isk}(\tau)\Big]\right| \\
&\le 2qC_2\rho_{\chi}^{|\ell|}\sum_{k=0}^{\infty}\rho_{\chi}^{2k}C_1 = \frac{2qC_2C_1\rho_{\chi}^{|\ell|}}{1-\rho_{\chi}^2} =: \mathcal{K}_1\rho_{\chi}^{|\ell|}, \text{ say.}
\end{aligned} \tag{A.1}
$$

Similarly, by Assumptions (A2), (B5), and (B7), for any $\ell \in \mathbb{Z}$,

$$
\begin{aligned}
\left|\frac{\mathrm{d}^2\gamma_{ij}^{\xi}(\tau;\ell)}{\mathrm{d}\tau^2}\right| &= \left|\sum_{k=0}^{\infty}\sum_{s=1}^{\infty}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left\{d_{isk}(\tau)d_{js,k+|\ell|}(\tau)\right\}\right| \\
&= \left|\sum_{k=0}^{\infty}\sum_{s=1}^{\infty}\Big[\frac{\mathrm{d}^2d_{isk}(\tau)}{\mathrm{d}\tau^2}d_{js,k+|\ell|}(\tau) + \frac{\mathrm{d}^2d_{js,k+|\ell|}(\tau)}{\mathrm{d}\tau^2}d_{isk}(\tau)\Big]\right| \\
&\le 2\rho_{\xi}^{|\ell|}\sum_{k=0}^{\infty}\rho_{\xi}^{2k}\sum_{s=1}^{\infty}B_{2is}B_{1js} = \frac{2B_2B_1\rho_{\xi}^{|\ell|}}{1-\rho_{\xi}^2} =: \mathcal{K}_2\rho_{\xi}^{|\ell|}, \text{ say.}
\end{aligned} \tag{A.2}
$$

Moreover, by Assumption (A3),

$$
\sigma_{ij}^{X}(\tau;\theta) = \sigma_{ij}^{\chi}(\tau;\theta) + \sigma_{ij}^{\xi}(\tau;\theta) = \frac{1}{2\pi}\sum_{\ell=-\infty}^{\infty}(\gamma_{ij}^{\chi}(\tau;\ell) + \gamma_{ij}^{\xi}(\tau;\ell))e^{-\iota\ell\theta}. \tag{A.3}
$$

Therefore, from (A.1) and (A.2) and noticing that $\mathcal{K}_1$ and $\mathcal{K}_2$ are independent of i, j, ℓ, and τ, we get

$$\max_{1\le i,j\le n}\sup_{\tau\in(0,1)}\sup_{\theta\in[-\pi,\pi]}\left|\frac{\mathrm{d}^2\sigma_{ij}^X(\tau;\theta)}{\mathrm{d}\tau^2}\right| \le \frac{1}{2\pi}\left\{\mathcal{K}_1\sum_{\ell=-\infty}^{\infty}\rho_\chi^{|\ell|}+\mathcal{K}_2\sum_{\ell=-\infty}^{\infty}\rho_\xi^{|\ell|}\right\}$$
$$\le \frac{1}{2\pi}\left\{\frac{2\mathcal{K}_1}{1-\rho_\chi}+\frac{2\mathcal{K}_2}{1-\rho_\xi}\right\}=:\mathcal{K},\text{ say,} \tag{A.4}$$

since $|e^{-\iota\theta\ell}|=1$ for all $\theta\in[-\pi,\pi]$ and all $\ell\in\mathbb{N}_0$. This proves part *(i)* of the lemma.

Then, because of Assumptions (A1) and (B1), for any $\ell\in\mathbb{Z}$,

$$|\gamma_{ij}^\chi(\tau;\ell)|=\left|\sum_{k=0}^{\infty}\sum_{s=1}^{q}c_{isk}(\tau)c_{js,k+|\ell|}(\tau)\right|\le\sum_{k=0}^{\infty}\sum_{s=1}^{q}C_1^2\rho_\chi^k\rho_\chi^{k+|\ell|}\le\frac{C_1^2\rho_\chi^{|\ell|}}{1-\rho_\chi^2}=:\mathcal{K}_3\rho_\chi^{|\ell|},\ \text{say.} \tag{A.5}$$

Similarly, because of Assumptions (A2) and (B5), for any $h\in\mathbb{Z}$,

$$|\gamma_{ij}^\xi(\tau;\ell)|=\left|\sum_{k=0}^{\infty}\sum_{s=1}^{\infty}d_{isk}(\tau)d_{js,k+|\ell|}(\tau)\right|\le\sum_{k=0}^{\infty}\sum_{s=1}^{\infty}B_{1is}B_{1js}\rho_\xi^k\rho_\xi^{k+|\ell|}\le\frac{B_1^2\rho_\xi^{|\ell|}}{1-\rho_\xi^2}=:\mathcal{K}_4\rho_\xi^{|\ell|},\text{ say.} \tag{A.6}$$

Therefore, using again (A.3), from (A.5) and (A.6) and noticing that $\mathcal{K}_3$ and $\mathcal{K}_4$ are independent of i, j, ℓ, and τ, we get

$$\max_{1\le i,j\le n}\sup_{\tau\in(0,1)}\sup_{\theta\in[-\pi,\pi]}\left|\frac{\mathrm{d}^2\sigma_{ij}^X(\tau;\theta)}{\mathrm{d}\theta^2}\right| \le \frac{1}{2\pi}\left\{\mathcal{K}_3\sum_{\ell=-\infty}^{\infty}|\ell|^2\rho_\chi^{|\ell|}+\mathcal{K}_4\sum_{\ell=-\infty}^{\infty}|\ell|^2\rho_\xi^{|\ell|}\right\}$$
$$=\frac{1}{2\pi}\left\{\frac{4\mathcal{K}_3}{(1-\rho_\chi)^3}+\frac{4\mathcal{K}_4}{(1-\rho_\xi)^3}\right\}=:\mathcal{K}',\ \text{say.} \tag{A.7}$$

This proves part *(ii)* of the lemma. □

A3 Proof of Lemma 2

Denote as $\sigma_{ij}^\xi(\tau;\theta)$ the generic (i,j) entry of $\mathbf{\Sigma}_n^\xi(\tau;\theta)$. For any $n\in\mathbb{N}_0$, $\tau\in(0,1)$, and $\theta\in[-\pi,\pi]$, we have

$$\lambda_{1;n}^\xi(\tau;\theta)=\|\mathbf{\Sigma}_n^\xi(\tau;\theta)\|\le\max_{1\le j\le n}\sum_{i=1}^{n}|\sigma_{ij}^\xi(\tau;\theta)|\le\frac{1}{2\pi}\max_{1\le j\le n}\sum_{i,\ell=1}^{n}|d_{i\ell}(\tau;e^{-\iota\theta})d_{\ell j}^\dagger(\tau;e^{-\iota\theta})|$$
$$\le\frac{1}{2\pi}\max_{1\le j\le n}\sum_{i,\ell=1}^{n}\sum_{k=0}^{\infty}|d_{i\ell k}(\tau)e^{-\iota\theta k}|\sum_{h=0}^{\infty}|d_{\ell jh}^\dagger(\tau)e^{\iota\theta h}|\le\frac{1}{2\pi}\max_{1\le j\le n}\sum_{i,\ell=1}^{n}\sum_{k=0}^{\infty}|d_{i\ell k}(\tau)|\sum_{h=0}^{\infty}|d_{\ell jh}^\dagger(\tau)|$$
$$\le\frac{1}{2\pi}\max_{1\le j\le n}\sum_{i,\ell=1}^{n}\sum_{k,h=0}^{\infty}B_{1i\ell}B_{1\ell j}\rho_\xi^k\rho_\xi^h\le\frac{B_1^2}{2\pi(1-\rho_\xi)^2},$$

because of Assumptions (A2) and (B5). Part *(i)* is proved by defining $B_\xi:=B_1^2/(2\pi(1-\rho_\xi)^2)$ and noting that it is independent of n, τ, and θ.

Parts *(ii)* and *(iii)* readily follow from Assumption (C) and part *(i)*, and an application of Weyl's inequality. □

A4 Proof of Lemma 3

Denote as $\lambda_{1;n}^\zeta(\tau;\theta)$ the largest eigenvalue of the spectral density of $\boldsymbol{\zeta}_{nt}(\tau)$. Then, for any $n\in\mathbb{N}_0$, $\tau\in(0,1)$, and $\theta\in[-\pi,\pi]$ (see also (15))

$$\lambda_{1;n}^\zeta(\tau;\theta)=\max_{\boldsymbol{a}\,:\,\boldsymbol{a}^\dagger\boldsymbol{a}=1}\boldsymbol{a}^\dagger\mathbf{A}_n(\tau;e^{-\iota\theta})\mathbf{\Sigma}_n^\xi(\tau;\theta)\mathbf{A}_n'(\tau;e^{\iota\theta})\boldsymbol{a}\le\lambda_{1;n}^\xi(\tau;\theta)\lambda_{1;n}^A(\tau;\theta) \tag{A.8}$$

where $\lambda^A_{1;n}(\tau;\theta)$ is the largest eigenvalue of $\mathbf{A}_n(\tau;e^{-\iota\theta})\mathbf{A}_n'(\tau;e^{\iota\theta})$. Moreover, denoting by $\lambda_1^{A^{(k)}}(\tau;\theta)$ the largest eigenvalue of $\mathbf{A}^{(k)}(\tau;e^{-\iota\theta})\mathbf{A}^{(k)\prime}(\tau;e^{\iota\theta})$ and recalling that $\mathbf{A}_n(\tau;L)$ is block-diagonal with diagonal blocks $\mathbf{A}^{(1)}(\tau;L),\dots,\mathbf{A}^{(m)}(\tau;L)$, we have

$$\lambda^A_{1;n}(\tau;\theta) \le \max_{1\le k\le n} \lambda_1^{A^{(k)}}(\tau;\theta) \le D_\zeta \tag{A.9}$$

where D_ζ is a constant independent of n, τ, and θ, because of Assumptions (D3) and (D4). By using Lemma 2 and (A.9) in (A.8), we have $\lambda^\zeta_{1;n}(\tau;\theta) \le B_\xi D_\zeta$. Therefore,

$$\mu^\zeta_{1;n}(\tau) = \max_{\boldsymbol{w}:\boldsymbol{w}'\boldsymbol{w}=1} \boldsymbol{w}'\boldsymbol{\Gamma}^\zeta_n(\tau)\boldsymbol{w} \le \int_{-\pi}^{\pi} \lambda^\zeta_{1;n}(\tau;\theta)\mathrm{d}\theta \le 2\pi B_\xi D_\zeta.$$

The proof is completed by defining $B_\zeta := 2\pi B_\xi D_\zeta$ and noting that it is independent of n, τ, and θ. □

A5 Proof of Lemma 4

The proof requires two intermediate results.

Lemma A1. *Under Assumptions (A) and (B) there exists a constant A_1 (independent of i and t) such that* $\sup_{\tau\in(0,1)} \mathrm{E}[|X_{it}(\tau)|^{r^*}] \le A_1$ *for all $i\in\mathbb{N}_0$ and $t\in\mathbb{Z}$, with r^* as defined in Assumption (A4).*

Proof of Lemma A1. By Minkowski inequality,

$$\sup_{\tau\in(0,1)} \left\{\mathrm{E}[|X_{it}(\tau)|^{r^*}]\right\}^{1/r^*} \le \sup_{\tau\in(0,1)} \left\{\mathrm{E}[|\chi_{it}(\tau)|^{r^*}]\right\}^{1/r^*} + \sup_{\tau\in(0,1)} \left\{\mathrm{E}[|\xi_{it}(\tau)|^{r^*}]\right\}^{1/r^*}. \tag{A.10}$$

Then, from (6)

$$\begin{aligned}\sup_{\tau\in(0,1)} \left\{\mathrm{E}[|\chi_{it}(\tau)|^{r^*}]\right\}^{1/r^*} &= \sup_{\tau\in(0,1)} \left\{\mathrm{E}\left[\left|\sum_{j=1}^{q}\sum_{k=0}^{\infty} c_{ijk}(\tau)u_{j,t-k}\right|^{r^*}\right]\right\}^{1/r^*}\\ &\le \sup_{\tau\in(0,1)} \sum_{j=1}^{q}\sum_{k=0}^{\infty} |c_{ijk}(\tau)| \left\{\mathrm{E}\left[|u_{j,t-k}|^{r^*}\right]\right\}^{1/r^*}\\ &\le C_0^{1/r^*} C_1 \sum_{k=0}^{\infty}\rho_\chi^k \le \frac{C_0^{1/r^*}C_1}{1-\rho_\chi} =: A_{11}, \text{ say,}\end{aligned} \tag{A.11}$$

because of Assumptions (A4) and (B1). Similarly, from (7),

$$\begin{aligned}\sup_{\tau\in(0,1)} \left\{\mathrm{E}[|\xi_{it}(\tau)|^{r^*}]\right\}^{1/r^*} &= \sup_{\tau\in(0,1)} \left\{\mathrm{E}\left[\left|\sum_{j=1}^{\infty}\sum_{k=0}^{\infty} d_{ijk}(\tau)\eta_{j,t-k}\right|^{r^*}\right]\right\}^{1/r^*}\\ &\le \sup_{\tau\in(0,1)} \sum_{j=1}^{\infty}\sum_{k=0}^{\infty} |d_{ijk}(\tau)| \left\{\mathrm{E}\left[|\eta_{j,t-k}|^{r^*}\right]\right\}^{1/r^*}\\ &\le C_0^{1/r^*} \sum_{j=1}^{\infty}\sum_{k=0}^{\infty}\rho_\xi^k B_{1ij} \le \frac{C_0^{1/r^*}B_1}{1-\rho_\xi} =: A_{12}, \text{ say,}\end{aligned} \tag{A.12}$$

because of Assumptions (A4) and (B5). Substituting (A.11) and (A.12) into (A.10) and defining $A_1 := A_{11} + A_{12}$ completes the proof. □

For all $t\in\mathbb{Z}$, let $\boldsymbol{\varepsilon} := \{\boldsymbol{\varepsilon}_t = (\mathbf{u}_t'\ \boldsymbol{\eta}_t')'\}$, and define $\mathcal{F}_t := (\dots,\boldsymbol{\varepsilon}_{t-1},\boldsymbol{\varepsilon}_t)$. Moreover, denoting by $\boldsymbol{\varepsilon}^* = \{\boldsymbol{\varepsilon}^*_t = (\mathbf{u}^{*\prime}_t\ \boldsymbol{\eta}^{*\prime}_t)'\}$ an independent copy of $\boldsymbol{\varepsilon}$, define $\mathcal{F}^*_t := (\dots,\boldsymbol{\varepsilon}_{-1},\boldsymbol{\varepsilon}^*_0,\boldsymbol{\varepsilon}_1,\dots,\boldsymbol{\varepsilon}_{t-1},\boldsymbol{\varepsilon}_t)$, which is a version of $\mathcal{F}_t$ where $\boldsymbol{\varepsilon}_0$ is replaced with $\boldsymbol{\varepsilon}^*_0$. Note that $\mathcal{F}^*_t = \mathcal{F}_t$ if $t<0$.

Then, from (6) and (7), it is clear that, for any $\tau \in (0,1)$, $i \in \mathbb{N}_0$, and $t \in \mathbb{Z}$, we have that $X_{it}(\tau) =: g_i(\tau; \mathcal{F}_t)$, where $g_i : (0,1) \times \mathbb{R}^\infty \to \mathbb{R}$ is a measurable function. Put $X_{it}^*(\tau) := g_i(\tau; \mathcal{F}_t^*)$. Then, for any $r > 0$, we define the physical dependence measure (see also Wu, 2005 and Zhang and Wu, 2019) as

$$\begin{aligned}\delta_{t,r,i} &= \sup_{\tau\in(0,1)} \{\mathrm{E}\left[(g_i(\tau;\mathcal{F}_t) - g_i(\tau;\mathcal{F}_t^*))^r\right]\}^{1/r} \\ &= \sup_{\tau\in(0,1)} \{\mathrm{E}\left[(X_{it}(\tau) - X_{it}^*(\tau))^r\right]\}^{1/r}, \ i \in \mathbb{N}_0, \ t \in \mathbb{Z}. \qquad \text{(A.13)}\end{aligned}$$

Lemma A2. *Under Assumptions (A) and (B) there exists a $\rho \in [0,1)$ and a constant A_2 (independent of r, i, and t) such that $\delta_{t,r,i} \leq A_2\rho^t$, $i \in \mathbb{N}_0$, and $t \in \mathbb{Z}$, for all $r \leq r^*$ where r^* is defined in Assumption (A4).*

Proof of Lemma A2. First, notice that

$$\begin{aligned}X_{it}(\tau) - X_{it}^*(\tau) &= \sum_{j=1}^{q} c_{ijt}(\tau)(u_{j0} - u_{j0}^*) + \sum_{j=1}^{\infty} d_{ijt}(\tau)(\eta_{j0} - \eta_{j0}^*) \\ &=: (\chi_{it}(\tau) - \chi_{it}^*(\tau)) + (\xi_{it}(\tau) - \xi_{it}^*(\tau)), \text{ say.} \qquad \text{(A.14)}\end{aligned}$$

By Minkowski inequality, it follows from (A.13) that

$$\delta_{t,r,i} \leq \sup_{\tau\in(0,1)} \{\mathrm{E}\left[(\chi_{it}(\tau) - \chi_{it}^*(\tau))^r\right]\}^{1/r} + \sup_{\tau\in(0,1)} \{\mathrm{E}\left[(\xi_{it}(\tau) - \xi_{it}^*(\tau))^r\right]\}^{1/r} =: \delta_{t,r,i}^\chi + \delta_{t,r,i}^\xi, \text{ say.} \qquad \text{(A.15)}$$

Then, from (A.14), for any $r \leq r^*$,

$$\begin{aligned}\delta_{t,r,i}^\chi &= \sup_{\tau\in(0,1)} \left\{\mathrm{E}\left[\left|\sum_{j=1}^{q} c_{ijt}(\tau)(u_{j0} - u_{j0}^*)\right|^r\right]\right\}^{1/r} \\ &\leq \sup_{\tau\in(0,1)} \sum_{j=1}^{q} |c_{ijt}(\tau)| \ \left\{\mathrm{E}\left[\left|(u_{j0} - u_{j0}^*)\right|^r\right]\right\}^{1/r} \leq 2C_0^{1/r} C_1 \rho_\chi^t, \qquad \text{(A.16)}\end{aligned}$$

because of Assumptions (A4) and (B1), and Minkowski inequality. Similarly, still from (A.14), for any $r \leq r^*$,

$$\begin{aligned}\delta_{t,r,i}^\xi &= \sup_{\tau\in(0,1)} \left\{\mathrm{E}\left[\left|\sum_{j=1}^{\infty} d_{ijt}(\tau)(\eta_{j0} - \eta_{j0}^*)\right|^r\right]\right\}^{1/r} \\ &\leq \sup_{\tau\in(0,1)} \sum_{j=1}^{\infty} |d_{ijt}(\tau)| \ \left\{\mathrm{E}\left[\left|(\eta_{j0} - \eta_{j0}^*)\right|^r\right]\right\}^{1/r} \leq 2C_0^{1/r} \sum_{j=1}^{\infty} B_{1ij}\rho_\xi^t \leq 2C_0^{1/r} B_1 \rho_\xi^t, \qquad \text{(A.17)}\end{aligned}$$

because of Assumptions (A4) and (B5), and Minkowski inequality. Substituting (A.16) and (A.17) into (A.15) and defining $\rho := \max(\rho_\chi, \rho_\xi)$ and $A_2 := 2C_0^{1/r}(C_1 + B_1)$ completes the proof. □

For any $n \in \mathbb{N}_0$, the mean-squared-error satisfies

$$\max_{1\leq i,j\leq n} \mathrm{E}\left[\sup_{\tau\in(0,1)} \sup_{\theta\in[-\pi,\pi]} \left|\widehat{\sigma}_{ij;T}^X(\tau;\theta) - \sigma_{ij}^X(\tau;\theta)\right|^2\right] \leq 2\left(\mathcal{V}_T + \Delta_T^2\right), \qquad \text{(A.18)}$$

where

$$\mathcal{V}_T := \max_{1\leq i,j\leq n} \mathrm{E}\left[\sup_{\tau\in(0,1)} \sup_{\theta\in[-\pi,\pi]} \left|\widehat{\sigma}_{ij;T}^X(\tau;\theta) - \mathrm{E}[\widehat{\sigma}_{ij;T}^X(\tau;\theta)]\right|^2\right] \quad \text{and} \qquad \text{(A.19)}$$

$$\Delta_T^2 := \max_{1\leq i,j\leq n} \sup_{\tau\in(0,1)} \sup_{\theta\in[-\pi,\pi]} \left\{\mathrm{E}[\widehat{\sigma}_{ij;T}^X(\tau;\theta)] - \sigma_{ij}^X(\tau;\theta)\right\}^2. \qquad \text{(A.20)}$$

First, let us consider (A.19). For any $\alpha > 0$ and $r > 0$, define the dependent adjusted norm

$$\Phi_{r,\alpha} := \max_{1\le i\le n} \sup_{k\in\mathbb{N}} (k+1)^\alpha \sum_{t=k}^{\infty} \delta_{t,r,i}$$

(see Zhang and Wu, 2019). For any $r \le r^*$,

$$\Phi_{r,\alpha} \le A_2 \sup_{k\in\mathbb{N}} (k+1)^\alpha \sum_{t=k}^{\infty} \rho^t = \frac{A_2}{1-\rho} \sup_{k\in\mathbb{N}} (k+1)^\alpha \rho^k = \frac{A_2}{1-\rho}, \tag{A.21}$$

because of Lemma A2. Moreover, because of Lemma A1, we can apply Corollary 4.4 in Zhang and Wu (2019) in the case $\alpha > 1/2 - 2/r^*$, which, along with (A.21), implies

$$\begin{aligned}\mathcal{V}_T &\le \Phi_{r,\alpha}^4 \left(\mathcal{K}_1 \frac{m_T \log T}{M_T} + \mathcal{K}_2 \frac{m_T^2 T^{4/r} (\log M_T)^{4+4/r}}{M_T^2} \right) \\ &\le \left(\frac{A_2}{1-\rho} \right)^4 \left(\mathcal{K}_1 \frac{m_T \log T}{M_T} + \mathcal{K}_2 \frac{m_T^2 T^{4/r} (\log M_T)^{4+4/r}}{M_T^2} \right), \end{aligned} \tag{A.22}$$

for some constants $\mathcal{K}_1$ and $\mathcal{K}_2$ (independent of T) and any $r \le r^*$. Recalling that $M_T = \lfloor T b_T \rfloor$ and $m_T = \lfloor 1/h_T \rfloor$, we thus obtain, for any $r \le r^*$,

$$\mathcal{V}_T \le C_X \frac{\log T}{T b_T h_T} + C_X' \frac{T^{4/r} (\log T)^{4+4/r}}{T^2 b_T^2 h_T^2} = \mathcal{A}_T + \mathcal{B}_{T,r} \tag{A.23}$$

for some constants C_X and C_X' (independent of T and n).

The following Lemma which is similar to Theorem 2.1 in Dahlhaus (2012) (see also Dahlhaus, 1996, Theorem 2.1), is needed to bound the bias.

Lemma A3. *Under Assumptions (A), (B), and (F), there exists constants $\mathcal{M}$, $\mathcal{L}$, and $\mathcal{H}$ (independent of i, j, ℓ and τ), such that*

$$\max_{1\le i,j\le n} \sup_{\tau\in(0,1)} \max_{|\ell|\le(M_T-1)} \left| \mathrm{E}[\widehat{\gamma}_{ij;T}^X(\tau;\ell)] - \gamma_{ij}^X(\tau;\ell) \right| \le \mathcal{M} b_T^2 + o(b_T^2) + \frac{\mathcal{L}}{T b_T} + \frac{\mathcal{H}}{T}. \tag{A.24}$$

Proof of Lemma A3. Consider (A.20). Let $\gamma_{ij}^X(\tau;\ell)$ and $\widehat{\gamma}_{ij;T}^X(\tau;\ell)$ be the (i,j) entries of the lag ℓ autocovariance matrix $\mathbf{\Gamma}_n^X(\tau;\ell)$ and its estimator $\widehat{\mathbf{\Gamma}}_n^X(\tau;\ell)$ as defined in (16), respectively. By Assumptions (A1), (B1), and (B3):

$$\begin{aligned}\left| \gamma_{ij}^\chi(\tau;\ell) - \mathrm{E}[\chi_{i,\lfloor\tau T\rfloor} \chi_{j,\lfloor\tau T\rfloor-\ell}'] \right| &\le \sum_{k=0}^{\infty} \sum_{s=1}^{q} \left| c_{isk}(\tau) c_{js,k+|\ell|}(\tau) - c_{isk}^*(\lfloor\tau T\rfloor) c_{js,k+|\ell|}^*(\lfloor\tau T\rfloor) \right| \\ &\le \frac{2q C_\chi C_1' \rho_\chi^{|\ell|}}{T} \sum_{k=0}^{\infty} \rho_\chi^{2k} + o(T^{-1}) \\ &\le \frac{2q C_\chi C_1'}{T(1-\rho_\chi^2)} + o(T^{-1}) =: \frac{\mathcal{H}_1}{T} + o(T^{-1}), \text{ say.}\end{aligned} \tag{A.25}$$

Similarly, by Assumptions (A2), (B5), and (B6),

$$\begin{aligned}\left| \gamma_{ij}^\xi(\tau;\ell) - \mathrm{E}[\xi_{i,\lfloor\tau T\rfloor} \xi_{j,\lfloor\tau T\rfloor-\ell}'] \right| &\le \sum_{k=0}^{\infty} \sum_{s=1}^{\infty} \left| d_{isk}(\tau) d_{js,k+|\ell|}(\tau) - d_{isk}^*(\lfloor\tau T\rfloor) d_{js,k+|\ell|}^*(\lfloor\tau T\rfloor) \right| \\ &\le \frac{2\rho_\xi^{|\ell|}}{T} \sum_{s=1}^{\infty} B_{\xi is} B_{1js}' \sum_{k=0}^{\infty} \rho_\chi^{2k} + o(T^{-1}) \\ &\le \frac{2 B_\xi B_1'}{T(1-\rho_\xi^2)} + o(T^{-1}) =: \frac{\mathcal{H}_2}{T} + o(T^{-1}), \text{ say.}\end{aligned} \tag{A.26}$$

Notice that, in (A.25) and (A.26), $\mathcal{H}_1$ and $\mathcal{H}_2$, as well as the remainders, are independent of i, j, ℓ, and τ. Therefore, from (A.25) and (A.26), because of Assumption (A3) (on the mutual uncorrelatedness of the common and idiosyncratic shocks) we have

$$\max_{1\le i,j\le n}\sup_{\tau\in(0,1)}\sup_{\ell\in\mathbb{Z}}\left|\gamma_{ij}^X(\tau;\ell)-\mathrm{E}[X_{i,\lfloor\tau T\rfloor}X'_{j,\lfloor\tau T\rfloor-\ell}]\right|\le\frac{\mathcal{H}_1}{T}+\frac{\mathcal{H}_2}{T}+o(T^{-1})=:\frac{\mathcal{H}}{T},\text{ say.}\tag{A.27}$$

This is the same result as in, for example, Dahlhaus (2012, equation (73)).

Then, for any $\tau\in(0,1)$ and $|\ell|\le(M_T-1)$, using (16) and (A.27),

$$\begin{aligned}\left|\mathrm{E}[\widehat{\gamma}_{ij;T}^X(\tau;\ell)]-\gamma_{ij}^X(\tau;\ell)\right|&=\left|\left\{\frac{1}{M_T}\sum_{s=T_1(\tau)+\ell}^{T_2(\tau)}\mathrm{J}\left(\frac{s-\lfloor\tau T\rfloor}{M_T}\right)\mathrm{E}[X_{i,s-\ell}X'_{js}]\right\}-\gamma_{ij}^X(\tau;\ell)\right|\\&\le\left|\left\{\frac{1}{M_T}\sum_{s=T_1(\tau)+\ell}^{T_2(\tau)}\mathrm{J}\left(\frac{s-\lfloor\tau T\rfloor}{M_T}\right)\gamma_{ij}^X(s/T;\ell)\right\}-\gamma_{ij}^X(\tau;\ell)\right|+\frac{\mathcal{H}}{T}.\end{aligned}\tag{A.28}$$

Moreover, by a Taylor expansion of order two of $\sigma_{ij}^X(s/T;\theta)$ in its first argument in a neighborhood of τ, we have, in view of Assumption (F1), for any $T_1(\tau)+\ell\le s\le T_2(\tau)$

$$\begin{aligned}\frac{1}{M_T}\sum_{s=T_1(\tau)+\ell}^{T_2(\tau)}\mathrm{J}\left(\frac{s-\lfloor\tau T\rfloor}{M_T}\right)\gamma_{ij}^X(s/T;\ell)&=\frac{1}{2\lfloor Tb_T\rfloor}\sum_{s=T_1(\tau)+\ell}^{T_2(\tau)}\mathrm{J}\left(\frac{s-\lfloor\tau T\rfloor}{2\lfloor Tb_T\rfloor}\right)\int_{-\pi}^{\pi}e^{\iota\theta\ell}\sigma_{ij}^X(s/T;\theta)\mathrm{d}\theta\\&\le\int_{-1/2}^{1/2}\mathrm{J}(u)\mathrm{d}u\int_{-\pi}^{\pi}e^{\iota\theta\ell}\sigma_{ij}^X(\tau;\theta)\mathrm{d}\theta+\frac{b_T^2}{2}\int_{-1/2}^{1/2}u^2\mathrm{J}(u)\mathrm{d}u\int_{-\pi}^{\pi}e^{\iota\theta\ell}\frac{\mathrm{d}^2\sigma_{ij}^X(\tau;\theta)}{\mathrm{d}\tau^2}\mathrm{d}\theta+o(b_T^2)+\frac{\mathcal{L}}{Tb_T}\\&=\gamma_{ij}^X(\tau;\ell)+\frac{b_T^2}{2}\int_{-1/2}^{1/2}u^2\mathrm{J}(u)\mathrm{d}u\int_{-\pi}^{\pi}e^{\iota\theta\ell}\frac{\mathrm{d}^2\sigma_{ij}^X(\tau;\theta)}{\mathrm{d}\tau^2}\mathrm{d}\theta+o(b_T^2)+\frac{\mathcal{L}}{Tb_T},\end{aligned}\tag{A.29}$$

where $\mathcal{L}$ and the remainders are independent of i, j, ℓ, and τ due to Assumptions (B4) and (B7). Notice also that the first-order term of the Taylor expansion of $\sigma_{ij}^X(s/T;\theta)$ drops out due to the symmetry of the kernel J about the origin.

Substituting (A.29) into (A.28), we get, in view of Lemma 1*(i)* (where $\mathcal{K}$ is defined),

$$\max_{1\le i,j\le n}\sup_{\tau\in(0,1)}\max_{|\ell|\le(M_T-1)}\left|\mathrm{E}[\widehat{\gamma}_{ij;T}^X(\tau;\ell)]-\gamma_{ij}^X(\tau;\ell)\right|\le b_T^2\frac{\mathcal{K}}{2}\int_{-1/2}^{1/2}u^2\mathrm{J}(u)\mathrm{d}u+o(b_T^2)+\frac{\mathcal{H}}{T}+\frac{\mathcal{L}}{Tb_T},$$

since $|e^{\iota\theta\ell}|=1$ for all $\theta\in[-\pi,\pi]$ and $\ell\in\mathbb{Z}$. Defining $\mathcal{M}:=\mathcal{K}/2\int_{-1/2}^{1/2}u^2\mathrm{J}(u)\mathrm{d}u$ (which is finite by Assumption (F1)) completes the proof. □

Because of Lemma A3, using in (17) the triangular kernel given in (24), for any $\theta\in[-\pi,\pi]$ and $\tau\in(0,1)$, the bias of our spectral estimator satisfies

$$\begin{aligned}2\pi\left\{\mathrm{E}[\widehat{\sigma}_{ij;T}^X(\tau;\theta)]-\sigma_{ij}^X(\tau;\theta)\right\}&=\sum_{\ell=-m_T}^{m_T}\left(1-\frac{|\ell|}{m_T}\right)\mathrm{E}[\widehat{\gamma}_{ij}^X(\tau;\ell)]e^{-\iota\ell\theta}-\sum_{\ell=-\infty}^{\infty}\gamma_{ij}^X(\tau;\ell)e^{-\iota\ell\theta}\\&=\sum_{\ell=-m_T}^{m_T}\left(1-\frac{|\ell|}{m_T}\right)\gamma_{ij}^X(\tau;\ell)e^{-\iota\ell\theta}-\sum_{\ell=-m_T}^{m_T}\gamma_{ij}^X(\tau;\ell)e^{-\iota\ell\theta}\\&\quad+\sum_{\ell=-m_T}^{m_T}\left(1-\frac{|\ell|}{m_T}\right)\left\{\mathrm{E}[\widehat{\gamma}_{ij}^X(\tau;\ell)]-\gamma_{ij}^X(\tau;\ell)\right\}e^{-\iota\ell\theta}-\sum_{|\ell|>m_T}\gamma_{ij}^X(\tau;\ell)e^{-\iota\ell\theta}\\&\le\sum_{\ell=-m_T}^{m_T}\frac{|\ell|}{m_T}|\gamma_{ij}^X(\tau;\ell)|+\sum_{\ell=-m_T}^{m_T}\left(1-\frac{|\ell|}{m_T}\right)\left(\mathcal{M}b_T^2+\frac{\mathcal{L}}{Tb_T}\right)+\sum_{|\ell|>m_T}|\gamma_{ij}^X(\tau;\ell)|+o(m_Tb_T^2)+\frac{m_T\mathcal{H}}{T}\\&=:\mathcal{S}_{1ijT}(\tau;\theta)+\mathcal{S}_{2ijT}(\tau;\theta)+\mathcal{S}_{3ijT}(\tau;\theta)+o(m_Tb_T^2)+o(m_TT^{-1}b_T^{-1}),\text{ say.}\end{aligned}\tag{A.30}$$

Then, from (A.5) and (A.6) in the proof of Lemma 1, because of Assumption (A3), we have

$$\max_{1\leq i,j\leq n}\sup_{\tau\in(0,1)}|\gamma^X_{ij}(\tau;\ell)|\leq \mathcal{K}_3\rho_\chi^{|\ell|}+\mathcal{K}_4\rho_\xi^{|\ell|}, \tag{A.31}$$

hence

$$\max_{1\leq i,j\leq n}\sup_{\tau\in(0,1)}\sup_{\theta\in[-\pi,\pi]}\mathcal{S}_{1ijT}(\tau;\theta)\leq\sum_{\ell=-\infty}^{\infty}\left\{\mathcal{K}_3\rho_\chi^{|\ell|}+\mathcal{K}_4\rho_\xi^{|\ell|}\right\}\frac{|\ell|}{m_T}\leq\frac{2\mathcal{K}_3\rho_\chi}{m_T(1-\rho_\chi)^2}+\frac{2\mathcal{K}_4\rho_\xi}{m_T(1-\rho_\xi)^2}. \tag{A.32}$$

Similarly, from Lemma A3,

$$\max_{1\leq i,j\leq n}\sup_{\tau\in(0,1)}\sup_{\theta\in[-\pi,\pi]}\mathcal{S}_{2ijT}(\tau;\theta)\leq\mathcal{M}m_Tb_T^2+\frac{\mathcal{L}}{Tb_T}m_T. \tag{A.33}$$

Finally,

$$\max_{1\leq i,j\leq n}\sup_{\tau\in(0,1)}\sup_{\theta\in[-\pi,\pi]}\mathcal{S}_{3ijT}(\tau;\theta)\leq\sum_{|\ell|>m_T}\left\{\mathcal{K}_3\rho_\chi^{|\ell|}+\mathcal{K}_4\rho_\xi^{|\ell|}\right\}\frac{|\ell|}{m_T}\leq\frac{F}{m_T}, \tag{A.34}$$

for some constant F (independent of i,j, ℓ, and τ). Substituting (A.32), (A.33), and (A.34) into (A.30), squaring the bias, and noticing that all those results are independent of i,j, τ, and θ, we obtain, from (A.20) (recall that $M_T=2\lfloor Tb_T\rfloor$ and $m_T=\lfloor 1/h_T\rfloor$),

$$\Delta_T^2\leq C''_X\left(h_T^2+\frac{b_T^4}{h_T^2}+\frac{1}{T^2b_T^2h_T^2}\right)\;+\;o\left(\frac{b_T^4}{h_T^2}\right)+o\left(\frac{1}{T^2b_T^2h_T^2}\right) \tag{A.35}$$

for some constant C''_X (independent of T and n).

Substituting (A.23) and (A.35) into (A.18) completes the proof. □

A6 Proof of Proposition 1

We divide the proof into four steps corresponding to the four steps of the estimation procedure.

(i) - Estimation of Spectral Density. Recall that, as discussed in Section 3.1, the estimator of the spectral density can be computed only for t/T with $M_T/2\leq t\leq(T-M_T/2)$ and for $\theta_j=\pi jh_T$ with $|j|\leq m_T$. For simplicity of notation, we let $\mathcal{T}_T:=\{M_T/2,\dots,(T-M_T/2)\}$. Then, a straightforward implication of Lemma 4 is that there exists a constant C^* (independent of T and n) such that, for any $n,T\in\mathbb{N}_0$,

$$\max_{1\leq i,k\leq n}\mathrm{E}\left[\max_{t\in\mathcal{T}_T}\max_{|j|\leq m_T}\left|\widehat{\sigma}^X_{ik;T}(t/T;\theta_j)-\sigma^X_{ik}(t/T;\theta_j)\right|^2\right]\leq C^*\zeta_{T,r^*} \tag{A.36}$$

with ζ_{T,r^*} defined in (26). From (A.36), for any $n\in\mathbb{N}_0$,

$$\begin{aligned}\mathrm{E}&\left[\max_{t\in\mathcal{T}_T}\max_{|j|\leq m_T}\frac{1}{n^2}\left\|\widehat{\boldsymbol{\Sigma}}^X_{n,T}(t/T;\theta_j)-\boldsymbol{\Sigma}^X_n(t/T;\theta_j)\right\|^2\right]\\&\leq\mathrm{E}\left[\max_{t\in\mathcal{T}_T}\max_{|j|\leq m_T}\frac{1}{n^2}\left\|\widehat{\boldsymbol{\Sigma}}^X_{n,T}(t/T;\theta_j)-\boldsymbol{\Sigma}^X_n(t/T;\theta_j)\right\|_F^2\right]\\&=\frac{1}{n^2}\mathrm{E}\left[\max_{t\in\mathcal{T}_T}\max_{|j|\leq m_T}\sum_{i=1}^n\sum_{k=1}^n\left|\widehat{\sigma}^X_{ik;T}(t/T;\theta_j)-\sigma^X_{ik}(t/T;\theta_j)\right|^2\right]\\&\leq\frac{1}{n^2}\sum_{i=1}^n\sum_{k=1}^n\mathrm{E}\left[\max_{t\in\mathcal{T}_T}\max_{|j|\leq m_T}\left|\widehat{\sigma}^X_{ik;T}(t/T;\theta_j)-\sigma^X_{ik}(t/T;\theta_j)\right|^2\right]\\&\leq C^*\zeta_{T,r^*}.\end{aligned}$$

Therefore, by Chebychev's inequality, for any $n \in \mathbb{N}_0$, as $T \to \infty$,

$$\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\frac{1}{n}\left\|\widehat{\mathbf{\Sigma}}^X_{n,T}(t/T;\theta_j)-\mathbf{\Sigma}^X_n(t/T;\theta_j)\right\| = O_{\mathrm{P}}\left(\zeta^{1/2}_{T,r^*}\right). \tag{A.37}$$

Let $\boldsymbol{\ell}_i$ denote the ith vector in the n-dimensional canonical basis, i.e. the vector with 1 in entry i and 0 elsewhere. Then, again from (A.36), for any $n \in \mathbb{N}_0$ and $1 \le i \le n$,

$$\begin{aligned}
&\mathrm{E}\left[\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\frac{1}{n}\left\|\boldsymbol{\ell}_i'\Big(\widehat{\mathbf{\Sigma}}^X_{n,T}(t/T;\theta_j)-\mathbf{\Sigma}^X_n(t/T;\theta_j)\Big)\right\|^2\right]\\
&= \mathrm{E}\left[\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\frac{1}{n}\boldsymbol{\ell}_i'\Big(\widehat{\mathbf{\Sigma}}^X_{n,T}(t/T;\theta_j)-\mathbf{\Sigma}^X_n(t/T;\theta_j)\Big)\Big(\widehat{\mathbf{\Sigma}}^X_{n,T}(t/T;\theta_j)-\mathbf{\Sigma}^X_n(t/T;\theta_j)\Big)'\boldsymbol{\ell}_i\right]\\
&= \mathrm{E}\left[\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\frac{1}{n}\boldsymbol{\ell}_i'\Big(\widehat{\mathbf{\Sigma}}^X_{n,T}(t/T;\theta_j)-\mathbf{\Sigma}^X_n(t/T;\theta_j)\Big)\Big(\widehat{\mathbf{\Sigma}}^X_{n,T}(t/T;\theta_j)-\mathbf{\Sigma}^X_n(t/T;\theta_j)\Big)\boldsymbol{\ell}_i\right]\\
&= \frac{1}{n}\mathrm{E}\left[\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\sum_{k=1}^n\left|\widehat{\sigma}^X_{ik;n,T}(t/T;\theta_j)-\sigma^X_{ik}(t/T;\theta_j)\right|^2\right]\\
&\le \frac{1}{n}\sum_{k=1}^n\mathrm{E}\left[\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\left|\widehat{\sigma}^X_{ik;n,T}(t/T;\theta_j)-\sigma^X_{ik}(t/T;\theta_j)\right|^2\right] \le C^*\zeta_{T,r^*},
\end{aligned} \tag{A.38}$$

where $\|\cdot\|$ in this case denotes the Euclidean norm of a vector. Therefore, by Chebychev's inequality, and since C^* in (A.38) does not depend on i, for any $\varepsilon > 0$, there exists $\eta(\varepsilon)$ and an integer $T^* = T(\varepsilon)$, both independent of i, such that

$$\mathrm{P}\left(\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\frac{1}{\sqrt{n\zeta_{T,r^*}}}\left\|\boldsymbol{\ell}_i'\Big(\widehat{\mathbf{\Sigma}}^X_{n,T}(t/T;\theta_j)-\mathbf{\Sigma}^X_n(t/T;\theta_j)\Big)\right\| \ge \eta(\varepsilon)\right) < \varepsilon, \tag{A.39}$$

for all $n \in \mathbb{N}_0$, $1 \le i \le n$, and $T \ge T^*$. Equivalently, hereafter, we say that

$$\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\frac{1}{\sqrt{n}}\left\|\boldsymbol{\ell}_i'\Big(\widehat{\mathbf{\Sigma}}^X_{n,T}(t/T;\theta_j)-\mathbf{\Sigma}^X_n(t/T;\theta_j)\Big)\right\| = O_{\mathrm{P}}\left(\zeta^{1/2}_{T,r^*}\right) \tag{A.40}$$

as $T \to \infty$, uniformly in i. This proves the analogue of Lemma 1*(i)* and 1*(ii)* in Forni et al. (2017).

(ii) - Dynamic Principal Components. By using (A.37) and Lemma 2*(i)*, for any $n \in \mathbb{N}_0$, we have

$$\begin{aligned}
&\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\frac{1}{n}\left\|\widehat{\mathbf{\Sigma}}^X_{n,T}(t/T;\theta_j)-\mathbf{\Sigma}^\chi_n(t/T;\theta_j)\right\|\\
&\qquad\le \max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\frac{1}{n}\left\|\widehat{\mathbf{\Sigma}}^X_{n,T}(t/T;\theta_j)-\mathbf{\Sigma}^X_n(t/T;\theta_j)\right\| + \max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\frac{1}{n}\left\|\mathbf{\Sigma}^\xi_n(t/T;\theta_j)\right\|\\
&\qquad= O_{\mathrm{P}}(\zeta^{1/2}_{T,r^*}) + O(n^{-1}) = O_{\mathrm{P}}\left(\max\left(\zeta^{1/2}_{T,r^*}, n^{-1}\right)\right),
\end{aligned} \tag{A.41}$$

as $T \to \infty$. Similarly, we can show that

$$\begin{aligned}
&\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\frac{1}{\sqrt{n}}\left\|\boldsymbol{\ell}_i'\Big(\widehat{\mathbf{\Sigma}}^X_{n,T}(t/T;\theta_j)-\mathbf{\Sigma}^\chi_n(t/T;\theta_j)\Big)\right\|\\
&\le \max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\frac{1}{\sqrt{n}}\left\|\boldsymbol{\ell}_i'\Big(\widehat{\mathbf{\Sigma}}^X_{n,T}(t/T;\theta_j)-\mathbf{\Sigma}^X_n(t/T;\theta_j)\Big)\right\| + \max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\frac{1}{\sqrt{n}}\left\|\boldsymbol{\ell}_i'\mathbf{\Sigma}^\xi_n(t/T;\theta_j)\right\|\\
&= O_{\mathrm{P}}\left(\max\left(\zeta^{1/2}_{T,r^*}, n^{-1/2}\right)\right)
\end{aligned} \tag{A.42}$$

as $T \to \infty$, uniformly in i, because of (A.40), while for the second term we have (recall that $\boldsymbol{\ell}_i'\boldsymbol{\ell}_i = 1$)

$$\begin{aligned}
\max_{1\le i\le n}\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\frac{1}{n}\left\|\boldsymbol{\ell}_i'\boldsymbol{\Sigma}_n^{\xi}(t/T;\theta_j)\right\|^2 &= \max_{1\le i\le n}\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\frac{1}{n}\boldsymbol{\ell}_i'\boldsymbol{\Sigma}_n^{\xi}(t/T;\theta_j)\boldsymbol{\Sigma}_n^{\xi}(t/T;\theta_j)\boldsymbol{\ell}_i\\
&\le \max_{\boldsymbol{w}:\boldsymbol{w}'\boldsymbol{w}=1}\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\frac{1}{n}\boldsymbol{w}'\boldsymbol{\Sigma}_n^{\xi}(t/T;\theta_j)\boldsymbol{\Sigma}_n^{\xi}(t/T;\theta_j)\boldsymbol{w}\\
&= \max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\frac{1}{n}\|\boldsymbol{\Sigma}_n^{\xi}(t/T;\theta_j)\|^2 = O(n^{-1}) \qquad \text{(A.43)}
\end{aligned}$$

by definition of the largest eigenvalue of a matrix and Lemma 2*(i)*. This proves the analogue of Lemma 1*(iii)* and *(iv)* in Forni et al. (2017).

It follows from (A.41) and Weyl's inequality that, for all $1 \le \ell \le q$, as $n, T \to \infty$,

$$\begin{aligned}
\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\frac{1}{n}\left|\widehat{\lambda}^X_{\ell;n,T}(t/T;\theta_j) - \lambda^{\chi}_{\ell;n}(t/T;\theta_j)\right| &\le \max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\frac{1}{n}\left\|\widehat{\boldsymbol{\Sigma}}^X_{n,T}(t/T;\theta_j) - \boldsymbol{\Sigma}^{\chi}_n(t/T;\theta_j)\right\|\\
&= O_{\mathrm{P}}\left(\max\left(\zeta_{T,r^*}^{1/2}, n^{-1}\right)\right). \qquad \text{(A.44)}
\end{aligned}$$

Let $\widehat{\boldsymbol{\Lambda}}^X_{n,T}(t/T;\theta_j)$ and $\boldsymbol{\Lambda}^{\chi}_n(t/T;\theta_j)$ be the $q\times q$ diagonal matrices with the q largest eigenvalues of $\widehat{\boldsymbol{\Sigma}}^X_{n,T}(t/T;\theta_j)$ and $\boldsymbol{\Sigma}^{\chi}_n(t/T;\theta_j)$, respectively. Then, since q is finite, (A.44) holds uniformly in ℓ and, as $n, T\to\infty$, we have

$$\begin{aligned}
&\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\left\|\left(n^{-1}\boldsymbol{\Lambda}^{\chi}_n(t/T;\theta_j) - n^{-1}\widehat{\boldsymbol{\Lambda}}^X_{n,T}(t/T;\theta_j)\right)\right\|^2\\
&\qquad\le \sum_{\ell=1}^{q}\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\frac{1}{n^2}\left(\widehat{\lambda}^X_{\ell;n,T}(t/T;\theta_j) - \lambda^{\chi}_{\ell;n}(t/T;\theta_j)\right)^2 = O_{\mathrm{P}}\left(\max\left(\zeta_{T,r^*}, n^{-2}\right)\right). \qquad \text{(A.45)}
\end{aligned}$$

From Assumption (C), as $n\to\infty$

$$\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\left\|n(\boldsymbol{\Lambda}^{\chi}_n(t/T;\theta_j))^{-1}\right\| = \max_{t\in\mathcal{T}_T}\max_{|j|\le m_T} n(\lambda^{\chi}_{q;n}(t/T;\theta_j))^{-1} \le D, \qquad \text{(A.46)}$$

with $D > 0$ independent of n. And from (A.45), (A.46), and Lemma 2*(ii)*, as $n, T\to\infty$,

$$\begin{aligned}
&\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\left\|n(\widehat{\boldsymbol{\Lambda}}^X_{n,T}(t/T;\theta_j))^{-1}\right\|\\
&\qquad\le \max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\left\|n\big((\widehat{\boldsymbol{\Lambda}}^X_{n,T}(t/T;\theta_j))^{-1} - (\boldsymbol{\Lambda}^{\chi}_n(t/T;\theta_j))^{-1}\big)\right\| + \max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\left\|n(\boldsymbol{\Lambda}^{\chi}_n(t/T;\theta_j))^{-1}\right\|\\
&\qquad= O_{\mathrm{P}}\left(\max\left(\zeta_{T,r^*}^{1/2}, n^{-1}\right)\right) + O(1) = O_{\mathrm{P}}(1). \qquad \text{(A.47)}
\end{aligned}$$

This proves the analogue of Lemma 2 in Forni et al. (2017).

Let $\widehat{\mathbf{P}}^X_{n,T}(t/T;\theta_j)$ be the $n\times q$ matrix having as columns the normalized eigenvectors of $\widehat{\boldsymbol{\Sigma}}^X_{n,T}(t/T;\theta_j)$ corresponding to its q largest eigenvalues. Let $\mathbf{P}^{\chi}_n(t/T;\theta_j)$ be the $n\times q$ matrix having as columns the normalized eigenvectors of $\boldsymbol{\Sigma}^{\chi}_{n,T}(t/T;\theta_j)$ corresponding to its q largest eigenvalues. By "normalized" we mean that the q columns $\mathbf{p}^{\chi}_{j;n}(t/T;\theta_j)$ of $\mathbf{P}^{\chi}_n(t/T;\theta_j)$ are such that $\mathbf{p}^{\chi\dagger}_{j;n}(t/T;\theta_j)\mathbf{p}^{\chi}_{j;n}(t/T;\theta_j) = 1$.

Now, by Assumption (C),

$$\max_{1\le\ell\le q}\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\left(\lambda^{\chi}_{\ell;n}(t/T;\theta_j)n^{-1}\right) \ge C, \qquad \text{(A.48)}$$

for some constant C independent of n. Then, by Theorem 2 and Corollary 2 in Yu et al. (2015), there exists a $q\times q$ complex diagonal matrix $\boldsymbol{\mathcal{J}}(t/T;\theta_j)$ with unit modulus entries, such that, as $n,T\to\infty$,

$$\begin{aligned}\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}&\left\|\widehat{\mathbf{P}}^X_{n,T}(t/T;\theta_j)-\mathbf{P}^\chi_n(t/T;\theta_j)\boldsymbol{\mathcal{J}}(t/T;\theta_j)\right\|\\ &\le \max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\frac{2^{3/2}\sqrt{q}\left\|\widehat{\boldsymbol{\Sigma}}^X_{n,T}(t/T;\theta_j)-\boldsymbol{\Sigma}^\chi_n(t/T;\theta_j)\right\|}{\lambda^\chi_{q;n}(t/T;\theta_j)}\\ &\le 2^{3/2}\sqrt{q}\,Cn^{-1}\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\left\|\widehat{\boldsymbol{\Sigma}}^X_{n,T}(t/T;\theta_j)-\boldsymbol{\Sigma}^\chi_n(t/T;\theta_j)\right\|=O_{\mathrm{P}}\left(\max\left(\zeta_{T,r^*}^{1/2},n^{-1}\right)\right),\end{aligned}\tag{A.49}$$

because of (A.41) and (A.48). Noting that $\|\mathbf{P}^\chi_n(t/T;\theta_j)\boldsymbol{\mathcal{J}}(t/T;\theta_j)\|=1$ and $\|\boldsymbol{\ell}_i'\boldsymbol{\Sigma}^\chi_n(t/T;\theta_j)\|=O(\sqrt{n})$, moreover, for all $t\in\mathcal{T}_T$ and $|j|\le m_T$, as $n,T\to\infty$,

$$\begin{aligned}\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}&\sqrt{n}\left\|\boldsymbol{\ell}_i'\left(\widehat{\mathbf{P}}^X_{n,T}(t/T;\theta_j)-\mathbf{P}^\chi_n(t/T;\theta_j)\boldsymbol{\mathcal{J}}(t/T;\theta_j)\right)\right\|\\ &=\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\frac{1}{\sqrt{n}}\Big\|\boldsymbol{\ell}_i'\Big\{\widehat{\boldsymbol{\Sigma}}^X_{n,T}(t/T;\theta_j)\widehat{\mathbf{P}}^X_{n,T}(t/T;\theta_j)\,n(\widehat{\boldsymbol{\Lambda}}^X_{n,T}(t/T;\theta_j))^{-1}\\ &\quad-\boldsymbol{\Sigma}^\chi_n(t/T;\theta_j)\mathbf{P}^\chi_n(t/T;\theta_j)\,n(\boldsymbol{\Lambda}^\chi_n(t/T;\theta_j))^{-1}\Big\}\Big\|\\ &\le\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\frac{1}{\sqrt{n}}\left\|\boldsymbol{\ell}_i'\left(\widehat{\boldsymbol{\Sigma}}^X_{n,T}(t/T;\theta_j)-\boldsymbol{\Sigma}^\chi_n(t/T;\theta_j)\right)\right\|\,\|n(\boldsymbol{\Lambda}^\chi_n(t/T;\theta_j))^{-1}\|\\ &\quad+\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\frac{1}{\sqrt{n}}\left\|\boldsymbol{\ell}_i'\boldsymbol{\Sigma}^\chi_n(t/T;\theta_j)\right\|\,\left\|n\left((\boldsymbol{\Lambda}^\chi_n(t/T;\theta_j))^{-1}-(\widehat{\boldsymbol{\Lambda}}^X_{n,T}(t/T;\theta_j))^{-1}\right)\right\|\\ &\quad+\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\frac{1}{\sqrt{n}}\left\|\boldsymbol{\ell}_i'\boldsymbol{\Sigma}^\chi_n(t/T;\theta_j)\right\|\,\|n(\boldsymbol{\Lambda}^\chi_n(t/T;\theta_j))^{-1}\|\,\left\|\widehat{\mathbf{P}}^X_{n,T}(t/T;\theta_j)-\mathbf{P}^\chi_n(t/T;\theta_j)\boldsymbol{\mathcal{J}}(t/T;\theta_j)\right\|\\ &\quad+o_{\mathrm{P}}\left(\max\left(\zeta_{T,r^*}^{1/2},n^{-1/2}\right)\right)=O_{\mathrm{P}}\left(\max\left(\zeta_{T,r^*}^{1/2},n^{-1/2}\right)\right),\end{aligned}\tag{A.50}$$

uniformly in i, because of (A.42), (A.45), (A.46), and (A.49).

Furthermore, because of Assumptions (A1) and (B1),

$$\max_{1\le i\le n}\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\sigma^\chi_{ii}(t/T;\theta_j)=\max_{1\le i\le n}\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\sum_{\ell=1}^q c_{i\ell}(t/T;e^{-\iota\theta_j})c^\dagger_{\ell i}(t/T;e^{-\iota\theta_j})\le\frac{C_1^2}{(1-\rho_\chi)^2}\tag{A.51}$$

and, since (B1) holds for all $i\in\mathbb{N}_0$, (A.51) is independent of n. Therefore, denoting by $p^\chi_{i\ell}(t/T;\theta_j)$ the (i,ℓ) entry of $\mathbf{P}^\chi_n(t/T;\theta_j)$, we also have

$$\max_{1\le i\le n}\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\sigma^\chi_{ii}(t/T;\theta_j)=\max_{1\le i\le n}\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\sum_{\ell=1}^q\left(\lambda^\chi_{\ell;n}(t/T;\theta_j)\right)\left|p^\chi_{i\ell}(t/T;\theta_j)\right|^2\le\frac{C_1^2}{(1-\rho_\chi)^2}.\tag{A.52}$$

Therefore, replacing (A.48) into (A.52), we see that for all $1\le\ell\le q$ we must have

$$\max_{1\le i\le n}\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\left(n\left|p^\chi_{i\ell}(t/T;\theta_j)\right|^2\right)\le A$$

where the constant A is also independent of n since the constants in (A.48) and (A.52) are. It follows that

$$\max_{1\le i\le n}\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\sqrt{n}\left\|\boldsymbol{\ell}_i'\mathbf{P}^\chi_n(t/T;\theta_j)\right\|\le M\tag{A.53}$$

for some constant M that does not depend on n.

Finally, since $\boldsymbol{\mathcal{J}}^\dagger(t/T;\theta_j)\boldsymbol{\mathcal{J}}(t/T;\theta_j)=\mathbf{I}_q$ and $\boldsymbol{\mathcal{J}}^\dagger(t/T;\theta_j)\boldsymbol{\Lambda}^X_n(t/T;\theta_j)\boldsymbol{\mathcal{J}}(t/T;\theta_j)=\boldsymbol{\Lambda}^X_n(t/T;\theta_j)$, then, for any $n \in \mathbb{N}_0$,

$$\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\left\|\boldsymbol{\ell}_i'\Big(\mathbf{P}_n^{\chi}(t/T;\theta_j)(\boldsymbol{\Lambda}_n^{\chi}(t/T;\theta_j))^{1/2}\boldsymbol{\mathcal{J}}(t/T;\theta_j)-\widehat{\mathbf{P}}_{n,T}^{X}(t/T;\theta_j)(\widehat{\boldsymbol{\Lambda}}_{n,T}^{X}(t/T;\theta_j))^{1/2}\Big)\right\|$$
$$\le \max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\left\|\boldsymbol{\ell}_i'\sqrt{n}\Big(\mathbf{P}_n^{\chi}(t/T;\theta_j)\boldsymbol{\mathcal{J}}(t/T;\theta_j)-\widehat{\mathbf{P}}_{n,T}^{X}(t/T;\theta_j)\Big)\left(n^{-1}\boldsymbol{\Lambda}_n^{\chi}(t/T;\theta_j)\right)^{1/2}\right\|$$
$$+\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\left\|\sqrt{n}\boldsymbol{\ell}_i'\widehat{\mathbf{P}}_{n,T}^{X}(t/T;\theta_j)\Big(n^{-1/2}\big(\boldsymbol{\Lambda}_n^{\chi}(t/T;\theta_j)\big)^{1/2}-n^{-1/2}\big(\widehat{\boldsymbol{\Lambda}}_{n,T}^{X}(t/T;\theta_j)\big)^{1/2}\Big)\right\|$$
$$=: I+II, \quad \text{say.} \tag{A.54}$$

It follows from (A.46) and (A.50) that I is $O_{\mathrm{P}}(\max(\zeta_{T,r^*}^{1/2}, n^{-1/2}))$ uniformly in i. For II, we have

$$II \le \max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\left\{\left\|\sqrt{n}\boldsymbol{\ell}_i'\widehat{\mathbf{P}}_{n,T}^{X}(t/T;\theta_j)\right\|\left\|\Big(n^{-1/2}\big(\boldsymbol{\Lambda}_n^{\chi}(t/T;\theta_j)\big)^{1/2}-n^{-1/2}\big(\widehat{\boldsymbol{\Lambda}}_{n,T}^{X}(t/T;\theta_j)\big)^{1/2}\Big)\right\|\right\}$$
$$=: \max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\left(II_a\times II_b\right), \quad \text{say.} \tag{A.55}$$

It follows from (A.45) that II_b in (A.55) is $O_{\mathrm{P}}(\max(\zeta_{T,r^*}^{1/2}, n^{-1})$ uniformly in t and j, while for II_a we have (recall that $\|\boldsymbol{\mathcal{J}}(t/T;\theta_j)\|=1$)

$$\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T} II_b \le \max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\left\|\sqrt{n}\boldsymbol{\ell}_i'\Big(\widehat{\mathbf{P}}_{n,T}^{X}(t/T;\theta_j)-\mathbf{P}_n^{\chi}(t/T;\theta_j)\boldsymbol{\mathcal{J}}(t/T;\theta_j)\Big)\right\|$$
$$+\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\left\|\sqrt{n}\boldsymbol{\ell}_i'\mathbf{P}_n^{\chi}(t/T;\theta_j)\boldsymbol{\mathcal{J}}(t/T;\theta_j)\right\|$$
$$=O_{\mathrm{P}}(\max(\zeta_{T,r^*}^{1/2}, n^{-1/2})+O(1), \tag{A.56}$$

uniformly in i, because of (A.53) and (A.50). Hence, $II=O_{\mathrm{P}}(\max(\zeta_{T,r^*}^{1/2}, n^{-1})$ uniformly in i, and therefore, from (A.54), as $n,T\to\infty$,

$$\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\left\|\boldsymbol{\ell}_i'\Big(\mathbf{P}_n^{\chi}(t/T;\theta_j)(\boldsymbol{\Lambda}_n^{\chi}(t/T;\theta_j))^{1/2}\boldsymbol{\mathcal{J}}(t/T;\theta_j)-\widehat{\mathbf{P}}_{n,T}^{X}(t/T;\theta_j)(\widehat{\boldsymbol{\Lambda}}_{n,T}^{X}(t/T;\theta_j))^{1/2}\Big)\right\|$$
$$=O_{\mathrm{P}}\left(\max\left(\zeta_{T,r^*}^{1/2}, n^{-1/2}\right)\right), \tag{A.57}$$

uniformly in i. This proves the analogue of Lemma 4 in Forni et al. (2017).

Hereafter, let $\vartheta_{n,T,r^*} := \max\left(\zeta_{T,r^*}, n^{-1}\right)$. The spectral density matrix of the common component has rank q for all $\theta\in[-\pi,\pi]$ and $\tau\in(0,1)$ because of Assumption (C); for any $n\in\mathbb{N}_0$, ot can be expressed as

$$\boldsymbol{\Sigma}_n^{\chi}(t/T;\theta_j)=\left\{\mathbf{P}_n^{\chi}(t/T;\theta_j)[\boldsymbol{\Lambda}_n^{\chi}(t/T;\theta_j)]^{1/2}\boldsymbol{\mathcal{J}}(t/T;\theta_j)\right\}\left\{\boldsymbol{\mathcal{J}}^{\dagger}(t/T;\theta_j)[\boldsymbol{\Lambda}_n^{\chi}(t/T;\theta_j)]^{1/2}\mathbf{P}_n^{\chi\dagger}(t/T;\theta_j)\right\}$$
$$=\mathbf{P}_n^{\chi}(t/T;\theta_j)\boldsymbol{\Lambda}_n^{\chi}(t/T;\theta_j)\mathbf{P}_n^{\chi\dagger}(t/T;\theta_j), \tag{A.58}$$

with entries $\sigma_{ik}^{\chi}(t/T;\theta_j)=\boldsymbol{\ell}_i'\boldsymbol{\Sigma}_n^{\chi}(t/T;\theta_j)\boldsymbol{\ell}_k$. The estimator of the spectral density matrix of the common component is obtained by principal component analysis as

$$\widehat{\boldsymbol{\Sigma}}_{n,T}^{\chi}(t/T;\theta_j):=\widehat{\mathbf{P}}_{n,T}^{X}(t/T;\theta_j)[\widehat{\boldsymbol{\Lambda}}_{n,T}^{X}(t/T;\theta_j)]^{1/2}[\widehat{\boldsymbol{\Lambda}}_{n,T}^{X}(t/T;\theta_j)]^{1/2}\widehat{\mathbf{P}}_{n,T}^{X\dagger}(t/T;\theta_j)$$
$$=\widehat{\mathbf{P}}_{n,T}^{X}(t/T;\theta_j)\widehat{\boldsymbol{\Lambda}}_{n,T}^{X}(t/T;\theta_j)\widehat{\mathbf{P}}_{n,T}^{X\dagger}(t/T;\theta_j) \tag{A.59}$$

with entries $\widehat{\sigma}_{ik;n,T}^{\chi}(t/T;\theta_j)=\boldsymbol{\ell}_i'\widehat{\boldsymbol{\Sigma}}_{n,T}^{\chi}(t/T;\theta_j)\boldsymbol{\ell}_k$. Then, by comparing (A.58) with (A.59) and because of (A.57), for any $\varepsilon>0$, there exists $\eta(\varepsilon)$, $T^*=T^*(\varepsilon)$, and $N^*=N^*(\varepsilon)$, all independent of i and k, such that

$$\mathrm{P}\left(\max_{t\in\mathcal{T}_T}\max_{|j|\le m_T}\frac{\left|\widehat{\sigma}_{ik;n,T}^{\chi}(t/T;\theta_j)-\sigma_{ik}^{\chi}(t/T;\theta_j)\right|}{\vartheta_{n,T,r^*}^{1/2}}\ge\eta(\varepsilon)\right)<\varepsilon$$

for all $n \geq N^*$ and $T \geq T^*$. Equivalently as $n, T \to \infty$,

$$\max_{t \in \mathcal{T}_T} \max_{|j| \leq m_T} \left| \widehat{\sigma}^\chi_{ik;n,T}(t/T;\theta_j) - \sigma^\chi_{ik}(t/T;\theta_j) \right| = \max_{t \in \mathcal{T}_T} \max_{|j| \leq m_T} \left\| \boldsymbol{\ell}_i' \left(\widehat{\boldsymbol{\Sigma}}^\chi_{n,T}(t/T;\theta_j) - \boldsymbol{\Sigma}^\chi_n(t/T;\theta_j) \right) \boldsymbol{\ell}_k \right\|$$
$$= O_{\mathrm{P}} \left(\max \left(\zeta^{1/2}_{T,r^*}, n^{-1/2} \right) \right), \tag{A.60}$$

uniformly in i and k. This proves the analogue of Proposition 7 in Forni et al. (2017).

The (i,k) entry of the estimated lag ℓ autocovariance matrix $\widehat{\boldsymbol{\Gamma}}^\chi_{n,T}(\tau;k)$ defined in (19) is

$$\widehat{\gamma}^\chi_{ik;n,T}(t/T;\ell) = \frac{2\pi}{2m_T+1} \sum_{j=-m_T}^{m_T} \widehat{\sigma}^\chi_{ik;n,T}(t/T;\theta_k) e^{\iota \ell \theta_j} \tag{A.61}$$

and, by definition of a lag ℓ autocovariance, its population counterpart satisfies

$$\gamma^\chi_{ik}(t/T;\ell) = \int_{-\pi}^{\pi} \sigma^\chi_{ik}(t/T;\theta) e^{\iota \ell \theta} \mathrm{d}\theta. \tag{A.62}$$

Therefore, for any given lag ℓ (putting $\theta_{-m_T-1} := -\pi$), we have

$$\begin{aligned}
&\max_{t \in \mathcal{T}_T} |\widehat{\gamma}^\chi_{ik;n,T}(t/T;\ell) - \gamma^\chi_{ik}(t/T;\ell)| \\
&\leq \max_{t \in \mathcal{T}_T} \frac{2\pi}{2m_T+1} \sum_{j=-m_T}^{m_T} \left| e^{\iota \ell \theta_j} \widehat{\sigma}^\chi_{ik;n,T}(t/T;\theta_j) - e^{\iota \ell \theta_j} \sigma^\chi_{ik}(t/T;\theta_j) \right| \\
&\quad + \max_{t \in \mathcal{T}_T} \left| \frac{2\pi}{2m_T+1} \sum_{j=-m_T}^{m_T} e^{\iota \ell \theta_j} \sigma^\chi_{ik}(t/T;\theta_j) - \int_{-\pi}^{\pi} e^{\iota \ell \theta} \sigma^\chi_{ik}(t/T;\theta) \mathrm{d}\theta \right| \\
&\leq \max_{t \in \mathcal{T}_T} \frac{2\pi}{2m_T+1} \sum_{j=-m_T}^{m_T} \left| \widehat{\sigma}^\chi_{ik;n,T}(t/T;\theta_j) - \sigma^\chi_{ik}(t/T;\theta_j) \right| \\
&\quad + \max_{t \in \mathcal{T}_T} \frac{2\pi}{2m_T+1} \sum_{j=-m_T}^{m_T} \max_{\theta_{j-1} \leq \theta \leq \theta_j} \left| e^{\iota \ell \theta_j} \sigma^\chi_{ik}(t/T;\theta_j) - e^{\iota \ell \theta} \sigma^\chi_{ik}(t/T;\theta) \right| \\
&\leq \max_{t \in \mathcal{T}_T} 2\pi \max_{|j| \leq m_T} \left| \widehat{\sigma}^\chi_{ik;n,T}(t/T;\theta_j) - \sigma^\chi_{ik}(t/T;\theta_j) \right| + \frac{2\pi}{2m_T+1} \frac{C_1^2}{(1-\rho_\chi)^2} \sum_{j=-m_T}^{m_T} \max_{\theta_{j-1} \leq \theta \leq \theta_j} \left| e^{\iota \ell \theta_j} - e^{\iota \ell \theta} \right| \\
&\quad + \max_{t \in \mathcal{T}_T} \frac{2\pi}{2m_T+1} \sum_{j=-m_T}^{m_T} \max_{\theta_{j-1} \leq \theta \leq \theta_j} |\sigma^\chi_{ik}(t/T;\theta_j) - \sigma^\chi_{ik}(t/T;\theta)| = O_{\mathrm{P}}(\vartheta^{1/2}_{n,T,r^*}) + O(m_T^{-1}),
\end{aligned} \tag{A.63}$$

as $n, T \to \infty$, uniformly in i and k. For proving (A.63) we used (A.60) for the first term on the right-hand side, Assumption (B1) and the fact that the exponential function has bounded variation for the second, and Lemma 1*(ii)* for the third, which implies that the spectral density is Lipschitz continuous in θ uniformly in t. Moreover, the last term on the right-hand side of (A.63) is dominated by the first one because of Assumptions (F2) and (F3). Summing up, for any $\ell \in \mathbb{Z}$,

$$\max_{t \in \mathcal{T}_T} \left| \widehat{\gamma}^\chi_{ik;n,T}(t/T;\ell) - \gamma^\chi_{ik}(t/T;\ell) \right| = O_{\mathrm{P}}(\vartheta^{1/2}_{n,T,r^*}), \tag{A.64}$$

as $n, T \to \infty$, uniformly in i and k. This extends Proposition 8 in Forni et al. (2017) to the time-varying case.

(iii) - VAR filtering. Assuming that n factorizes, for some integer m, into $n = m(q+1)$, we estimate via Yule-Walker m distinct $(q+1)$-dimensional VAR models of order at most S (in view of Assumption

(D2)). For the sake of simplicity, let us assume $S = 1$: the Yule-Walker estimators of the VAR(1) coefficients (see also (20)) then are

$$\widehat{\mathbf{A}}^{(k)}_{n,T}(t/T) = \widehat{\mathbf{\Gamma}}^{\chi^{(k)}}_{n,T}(t/T;1)\left[\widehat{\mathbf{\Gamma}}^{\chi^{(k)}}_{n,T}(t/T;0)\right]^{-1}, \qquad 1 \leq k \leq m,$$

where $\widehat{\mathbf{\Gamma}}^{\chi^{(k)}}_{n,T}(t/T;\ell)$ is the $(q+1)\times(q+1)$ sub-matrix of $\widehat{\mathbf{\Gamma}}^{\chi}_{n,T}(t/T;\ell)$ corresponding to the lag ℓ autocovariance matrix of the sub-vector $\boldsymbol{\chi}^{(k)}_{n,T;t/T}$.

Since $\widehat{\mathbf{\Gamma}}^{\chi^{(k)}}_{n,T}(t/T;0)$ is finite-dimensional, (A.64) implies that it is consistent uniformly in t; together with Assumption (D4), it also implies that $\det[\widehat{\mathbf{\Gamma}}^{\chi^{(k)}}_{n,T}(t/T;0)] > d/2$, uniformly in t, with probability arbitrarily close to one for T large enough. The same arguments as in Appendix C of Forni et al. (2017) and (A.64) then entail, as $n,T\to\infty$,

$$\max_{t\in\mathcal{T}_T}\max_{1\leq k\leq m}\left\|\widehat{\mathbf{A}}^{(k)}_{n,T}(t/T) - \mathbf{A}^{(k)}_{n}(t/T)\right\| = O_{\mathrm{P}}(\vartheta^{1/2}_{n,T,r^*}). \tag{A.65}$$

Moreover, denoting by $\widehat{\mathbf{A}}_{n,T}(t/T)$ the $n\times n$ block-diagonal matrix having diagonal blocks $\widehat{\mathbf{A}}^{(k)}_{n,T}(t/T)$ for $1\leq k\leq m$, we have, from (A.65),

$$\max_{t\in\mathcal{T}_T}\frac{1}{\sqrt{n}}\left\|\widehat{\mathbf{A}}_{n,T}(t/T) - \mathbf{A}_{n}(t/T)\right\| = O_{\mathrm{P}}(\vartheta^{1/2}_{n,T,r^*}) \tag{A.66}$$

and

$$\max_{t\in\mathcal{T}_T}\left\|\boldsymbol{\ell}_i'\left(\widehat{\mathbf{A}}_{n,T}(t/T) - \mathbf{A}_{n}(t/T)\right)\right\| = O_{\mathrm{P}}(\vartheta^{1/2}_{n,T,r^*}) \tag{A.67}$$

as $n,T\to\infty$, uniformly in i, since by construction $\mathbf{A}_n(t/T)$ has only $m(q+1)^2$ non-zero entries. This extends Proposition 9 in Forni et al. (2017) to the time-varying setting.

We now establish the following two lemmas.

Lemma A4. *Under Assumptions (A) and (B), for any $n\in\mathbb{N}_0$,*
(i) $\max_{t\in\mathcal{T}_T}\max_{1\leq i\leq n}|X_{it}| = O_{\mathrm{P}}(\log^{1/\varphi} T)$ *and*
(ii) $\max_{t\in\mathcal{T}_T}\max_{1\leq i\leq n}|X_{it}(t/T)| = O_{\mathrm{P}}(\log^{1/\varphi} T)$
where φ is defined in Assumption (A5).

Proof. First notice that, because of Assumption (A5), for any $\varepsilon > 0$,

$$\mathrm{P}\left(\max_{t\in\mathcal{T}_T}\max_{1\leq j\leq q}|u_{jt}| > \varepsilon\right) \leq Tq\mathrm{P}\left(|u_{jt}|>\varepsilon\right) \leq K_u Tq\exp\left(-\varepsilon^{\varphi}K_u\right).$$

Hence, $\max_{t\in\mathcal{T}_T}\max_{1\leq j\leq q}|u_{jt}| = O_{\mathrm{P}}(\log^{1/\varphi} T)$. Likewise, since we assume $n = O(T^{\omega})$ for some $\omega > 0$, then $\max_{t\in\mathcal{T}_T}\max_{1\leq j\leq n}|\eta_{jt}| = O_{\mathrm{P}}(\log^{1/\varphi} T)$. The proof then follows from the absolute summability of the coefficients in (2), (3), (6), and (7) due to Assumptions (B1) and (B5). □

Lemma A5. *Under Assumptions (A) and (B), for any $n\in\mathbb{N}_0$,*

$$\max_{t\in\mathcal{T}_T}\max_{1\leq i\leq n}|X_{it} - X_{it}(t/T)| = O_{\mathrm{P}}(T^{-1}\log^{1/\varphi} T).$$

Proof. First let us show that

$$\max_{t\in\mathcal{T}_T}\max_{1\leq i\leq n}|\chi_{it} - \chi_{it}(t/T)| = O_{\mathrm{P}}(T^{-1}\log^{1/\varphi} T). \tag{A.68}$$

Without loss of generality, let us assume $q = 1$. From (2) and (6), for any $1\leq i\leq n$ and $K\geq 0$,

$$|\chi_{it} - \chi_{it}(t/T)| \leq \sum_{k=0}^{K}|c^*_{i1k}(t) - c_{i1k}(t/T)|\,|u_{t-k}| + \left|\sum_{k=K+1}^{\infty}(c^*_{i1k}(t) - c_{i1k}(t/T))u_{t-k}\right|.$$

Assumption (B1) implies that, for any $\varepsilon > 0$ and $\eta > 0$, there exists a constant $K^* = K(\varepsilon, \eta)$ independent of i, t, and T, such that

$$\mathrm{P}\left[\left|\sum_{k=K^*+1}^{\infty}(c^*_{i1k}(t) - c_{i1k}(t/T))u_{t-k}\right| > \eta/2\right] \leq \varepsilon/2.$$

Hence,

$$\begin{aligned}\mathrm{P}\left[|\chi_{it} - \chi_{it}(t/T)| > \eta\right] \leq &\, \mathrm{P}\left[\sum_{k=0}^{K^*}|c^*_{i1k}(t) - c_{i1k}(t/T)|\,|u_{t-k}| > \eta/2\right] \\ &+ \mathrm{P}\left[\left|\sum_{k=K^*+1}^{\infty}(c^*_{i1k}(t) - c_{i1k}(t/T))u_{t-k}\right| > \eta/2\right] \\ \leq &\, \mathrm{P}\left[\sum_{k=0}^{K^*}|c^*_{i1k}(t) - c_{i1k}(t/T)|\,|u_{t-k}| > \eta/2\right] + \varepsilon/2. \end{aligned} \tag{A.69}$$

Now, from Assumption (B3), since $\rho_\chi < 1$,

$$\mathrm{P}\left[\sum_{k=0}^{K^*}|c^*_{i1k}(t) - c_{i1k}(t/T)|\,|u_{t-k}| > \eta/2\right] \leq \mathrm{P}\left[\frac{K^* C_\chi}{T}\max_{t\in\mathcal{T}_T}|u_t| > \eta/2\right]$$

where (see the proof of Lemma A4) $\max_{t\in\mathcal{T}_T}|u_t| = O_{\mathrm{P}}(\log^{1/\varphi} T)$. It follows that there exists $T^* = T(\varepsilon, \eta)$ independent of i and t such that

$$\mathrm{P}\left[\sum_{k=0}^{K^*}|c^*_{i1k}(t) - c_{i1k}(t/T)|\,|u_{t-k}| > \eta/2\right] \leq \varepsilon/2 \tag{A.70}$$

for all $T \geq T^*$; (A.68) follows from putting together (A.69) and (A.70). The proof of

$$\max_{t\in\mathcal{T}_T}\max_{1\leq i\leq n}|\xi_{it} - \xi_{it}(t/T)| = O_{\mathrm{P}}(T^{-1}\log^{1/\varphi} T)$$

follows along the same steps, using Assumption (B6). The claim follows. □

(iv) - Principal Component Analysis. Since, for simplicity, we assumed $S = 1$ in (21),

$$\widehat{\mathbf{Z}}_{nt}(t/T) = \left[\mathbf{I}_n - \widehat{\mathbf{A}}_{n,T}(t/T)L\right]\mathbf{X}_{nt}, \qquad t \in \mathcal{T}_T. \tag{A.71}$$

Defining

$$\widetilde{\mathbf{Z}}_{nt}(t/T) := \left[\mathbf{I}_n - \mathbf{A}_n(t/T)L\right]\mathbf{X}_{nt}, \qquad t \in \mathcal{T}_T, \tag{A.72}$$

it follows from (A.66) and Lemma A4 that, as $n, T \to \infty$ (note that the filters in (A.71) and (A.72) just load $(q+1)$ series at a time)

$$\max_{t\in\mathcal{T}_T}\frac{1}{\sqrt{n}}\left\|\widehat{\mathbf{Z}}_{nt}(t/T) - \widetilde{\mathbf{Z}}_{nt}(t/T)\right\| = O_{\mathrm{P}}(\vartheta^{1/2}_{n,T,r^*}\log^{1/\varphi} T). \tag{A.73}$$

Lemma A5 moreover implies that, as $n, T \to \infty$,

$$\max_{t\in\mathcal{T}_T}\frac{1}{\sqrt{n}}\left\|\widetilde{\mathbf{Z}}_{nt}(t/T) - \mathbf{Z}_{nt}(t/T)\right\| = O_{\mathrm{P}}(T^{-1}\log^{1/\varphi} T). \tag{A.74}$$

By combining (A.73) and (A.74), as $n, T \to \infty$, we get

$$\max_{t\in\mathcal{T}_T}\frac{1}{\sqrt{n}}\left\|\widehat{\mathbf{Z}}_{nt}(t/T) - \mathbf{Z}_{nt}(t/T)\right\| = O_{\mathrm{P}}(\vartheta^{1/2}_{n,T,r^*}\log^{1/\varphi} T). \tag{A.75}$$

Similarly, from (A.67) and Lemma A5,

$$\max_{t\in\mathcal{T}_T}\left\|\boldsymbol{\ell}_i'\left(\widehat{\mathbf{Z}}_{nt}(t/T)-\mathbf{Z}_{nt}(t/T)\right)\right\| = O_{\mathrm{P}}(\vartheta_{n,T,r^*}^{1/2}\log^{1/\varphi}T), \tag{A.76}$$

uniformly in i.

Next, consider the rolling estimator

$$\widehat{\boldsymbol{\Gamma}}_{n,T}^{Z}(t/T) := \frac{1}{M_T}\sum_{s=T_1(t/T)}^{T_2(t/T)}\mathbf{Z}_{ns}(t/T)\mathbf{Z}_{ns}'(t/T), \quad t\in\mathcal{T}_T, \tag{A.77}$$

based on the uniform kernel as in (24) (see also (22)) of the covariance matrix of the unobservable $\mathbf{Z}_{nt}(t/T)$ and similarly define the estimator

$$\widehat{\boldsymbol{\Gamma}}_{n,T}^{\widehat{Z}}(t/T) := \frac{1}{M_T}\sum_{s=T_1(t/T)}^{T_2(t/T)}\widehat{\mathbf{Z}}_{ns}(t/T)\widehat{\mathbf{Z}}_{ns}'(t/T), \quad t\in\mathcal{T}_T \tag{A.78}$$

of the covariance matrix of the estimated $\widehat{\mathbf{Z}}_{nt}(t/T)$. Comparing (A.77) with (A.78), we obtain

$$\frac{1}{n}\left\|\widehat{\boldsymbol{\Gamma}}_{n,T}^{\widehat{Z}}(t/T)-\widehat{\boldsymbol{\Gamma}}_{n,T}^{Z}(t/T)\right\| = \frac{1}{nM_T}\left\|\sum_{s=T_1(t/T)}^{T_2(t/T)}\left[\widehat{\mathbf{Z}}_{ns}(t/T)\widehat{\mathbf{Z}}_{ns}'(t/T)-\mathbf{Z}_{ns}(t/T)\mathbf{Z}_{ns}'(t/T)\right]\right\|. \tag{A.79}$$

By (A.75), the right-hand side of (A.79) can be bounded uniformly in $t\in\mathcal{T}_T$, so that, as $n,T\to\infty$,

$$\max_{t\in\mathcal{T}_T}\frac{1}{n}\left\|\widehat{\boldsymbol{\Gamma}}_{n,T}^{\widehat{Z}}(t/T)-\widehat{\boldsymbol{\Gamma}}_{n,T}^{Z}(t/T)\right\| = O_{\mathrm{P}}(\vartheta_{n,T,r^*}^{1/2}\log^{1/\varphi}T). \tag{A.80}$$

and similarly, by (A.67) and (A.76),

$$\max_{t\in\mathcal{T}_T}\frac{1}{\sqrt{n}}\left\|\boldsymbol{\ell}_i'\left(\widehat{\boldsymbol{\Gamma}}_{n,T}^{\widehat{Z}}(t/T)-\widehat{\boldsymbol{\Gamma}}_{n,T}^{Z}(t/T)\right)\right\| = O_{\mathrm{P}}(\vartheta_{n,T,r^*}^{1/2}\log^{1/\varphi}T), \tag{A.81}$$

uniformly in i. Now, let $\boldsymbol{\Gamma}_n^Z(t/T)$ be the time-varying covariance matrix of the filtered process $\mathbf{Z}_n(t/T)$ obtained from $\mathbf{X}_n(t/T)$ as defined in (5) and (6), with (i,j) entry $\gamma_{ij}^Z(t/T)$ and denote as $\widehat{\gamma}_{ij;T}^Z(t/T)$ the (i,j) entry of $\widehat{\boldsymbol{\Gamma}}_{n,T}^Z(t/T)$. Starting from Corollary 3.4 in Zhang and Wu (2019), because of Lemmas A1 and A2, we can follow the same steps as those used by those authors for the spectral density estimation and leading from Corollary 4.4 to Lemma 4 in that paper. As a result, it is possible to show that there exists a constant C^{**} (independent of T and n) such that, for any $n,T\in\mathbb{N}_0$ (recalling that $M_T=2\lfloor Tb_T\rfloor$),

$$\max_{1\le i,j\le n}\mathrm{E}\left[\sup_{\tau\in(0,1)}\left|\widehat{\gamma}_{ij;T}^Z(\tau)-\gamma_{ij}^Z(\tau)\right|^2\right]\le C^{**}\psi_{T,r^*}, \tag{A.82}$$

where

$$\psi_{T,r^*} := \max\left(\frac{\log T}{Tb_T}, \frac{T^{4/r^*}(\log T)^{4+4/r^*}}{T^2b_T^2}, b_T^4\right).$$

Note that ψ_{T,r^*} is the maximum of three quantities. The first of them also appears in Rodríguez-Poo and Linton (2001, Proposition 3.2) and Motta et al. (2011, Theorem 1), while the third one is the square of the first quantity in (A.24).

Therefore, from (A.82), for any $n \in \mathbb{N}_0$,

$$\begin{aligned}
\mathrm{E}\left[\max_{t\in\mathcal{T}_T}\frac{1}{n^2}\left\|\widehat{\mathbf{\Gamma}}^{Z}_{n,T}(t/T)-\mathbf{\Gamma}^{Z}_{n}(t/T)\right\|^2\right] &\le \mathrm{E}\left[\max_{t\in\mathcal{T}_T}\frac{1}{n^2}\left\|\widehat{\mathbf{\Gamma}}^{Z}_{n,T}(t/T)-\mathbf{\Gamma}^{Z}_{n}(t/T)\right\|^2_F\right]\\
&= \frac{1}{n^2}\mathrm{E}\left[\max_{t\in\mathcal{T}_T}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\widehat{\gamma}^{Z}_{ij;T}(t/T)-\gamma^{Z}_{ij}(t/T)\right|^2\right]\\
&\le \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathrm{E}\left[\max_{t\in\mathcal{T}_T}\left|\widehat{\gamma}^{Z}_{ij;T}(t/T)-\gamma^{Z}_{ij}(t/T)\right|^2\right]\\
&\le C^{**}\psi_{T,r^*}.
\end{aligned}$$

Thus, by Chebychev's inequality and since $\psi_{T,r^*}=o(\zeta_{T,r^*})$ as $T\to\infty$,

$$\max_{t\in\mathcal{T}_T}\frac{1}{n}\left\|\widehat{\mathbf{\Gamma}}^{Z}_{n,T}(t/T)-\mathbf{\Gamma}^{Z}_{n}(t/T)\right\| = O_{\mathrm{P}}(\zeta^{1/2}_{T,r^*}). \tag{A.83}$$

Following similar steps as for (A.38), we also have

$$\max_{t\in\mathcal{T}_T}\frac{1}{\sqrt{n}}\left\|\boldsymbol{\ell}'_i\left(\widehat{\mathbf{\Gamma}}^{Z}_{n,T}(t/T)-\mathbf{\Gamma}^{Z}_{n}(t/T)\right)\right\| = O_{\mathrm{P}}(\zeta^{1/2}_{T,r^*}), \tag{A.84}$$

uniformly in i. Therefore, from (A.80), (A.81), (A.83), and (A.84),

$$\max_{t\in\mathcal{T}_T}\frac{1}{n}\left\|\widehat{\mathbf{\Gamma}}^{\widehat{Z}}_{n,T}(t/T)-\mathbf{\Gamma}^{Z}_{n,T}(t/T)\right\| = O_{\mathrm{P}}(\vartheta^{1/2}_{n,T,r^*}\log^{1/\varphi}T) \tag{A.85}$$

and

$$\max_{t\in\mathcal{T}_T}\frac{1}{\sqrt{n}}\left\|\boldsymbol{\ell}'_i\left(\widehat{\mathbf{\Gamma}}^{\widehat{Z}}_{n,T}(t/T)-\mathbf{\Gamma}^{Z}_{n,T}(t/T)\right)\right\| = O_{\mathrm{P}}(\vartheta^{1/2}_{n,T,r^*}\log^{1/\varphi}T), \tag{A.86}$$

as $n,T\to\infty$, uniformly in i. This proves the analogue of Lemma 11 in Forni et al. (2017).

Now, while we used (A.37) and (A.40) to estimate the dynamic model (8)-(9) via dynamic principal components, estimating the static model (15) via principal components can be achieved using (A.85) and (A.86). In particular, following similar steps as those leading to (A.41) and (A.42) and in view of Assumption (E) and Lemma 3, we can prove that

$$\max_{t\in\mathcal{T}_T}\frac{1}{n}\left\|\widehat{\mathbf{\Gamma}}^{\widehat{Z}}_{n,T}(t/T)-\mathbf{\Gamma}^{\psi}(t/T)\right\| = O_{\mathrm{P}}(\vartheta^{1/2}_{n,T,r^*}\log^{1/\varphi}T)$$

and

$$\max_{t\in\mathcal{T}_T}\frac{1}{\sqrt{n}}\left\|\boldsymbol{\ell}'_i\left(\widehat{\mathbf{\Gamma}}^{\widehat{Z}}_{n,T}(t/T)-\mathbf{\Gamma}^{\psi}(t/T)\right)\right\| = O_{\mathrm{P}}(\vartheta^{1/2}_{n,T,r^*}\log^{1/\varphi}T)$$

as $n,T\to\infty$, uniformly in i.

Then, let $\mathbf{M}^{\psi}(t/T)$ be the $q\times q$ diagonal matrix with the q largest eigenvalues of $\mathbf{\Gamma}^{\psi}_n(t/T)$ and $\mathbf{V}^{\psi}_n(t/T)$ the $n\times q$ matrix of the corresponding normalized eigenvectors. Similarly let $\widehat{\mathbf{M}}^{\widehat{Z}}_{n,T}(t/T)$ be the $q\times q$ diagonal matrix with entries the q largest eigenvalues of $\widehat{\mathbf{\Gamma}}^{Z}_{n,T}(t/T)$ and $\widehat{\mathbf{V}}^{\widehat{Z}}_{n,T}(t/T)$ the $n\times q$ matrix of the corresponding normalized eigenvectors. Following similar arguments as those leading to (A.45) and (A.50), we have

$$\max_{t\in\mathcal{T}_T}\frac{1}{n}\left\|\widehat{\mathbf{M}}^{\widehat{Z}}_{n,T}(t/T)-\mathbf{M}^{\psi}(t/T)\right\| = O_{\mathrm{P}}(\vartheta^{1/2}_{n,T,r^*}\log^{1/\varphi}T) \tag{A.87}$$

and

$$\max_{t\in\mathcal{T}_T}\sqrt{n}\left\|\boldsymbol{\ell}'_i\left(\widehat{\mathbf{V}}^{\widehat{Z}}_{n,T}(t/T)-\mathbf{V}^{\psi}(t/T)\mathbf{S}(t/T)\right)\right\| = O_{\mathrm{P}}(\vartheta^{1/2}_{n,T,r^*}\log^{1/\varphi}T) \tag{A.88}$$

as $n, T \to \infty$, uniformly in i, where $\mathbf{S}(t/T)$ is a $q \times q$ diagonal matrix with entries $s_j(t) = \pm 1$.

Now, by Assumption (A1), we have $\mathrm{E}[\mathbf{u}_t \mathbf{u}_t'] = \mathbf{I}_q$. Therefore, for all $\tau \in (0,1)$, the common component of the static factor model (15) has covariance $\mathbf{\Gamma}_n^{\psi}(\tau) = \mathbf{R}_n(\tau)\mathbf{R}_n'(\tau)$ and, by construction, we have $\mathbf{R}_n(\tau) := \mathbf{V}_n^{\psi}(\tau)[\mathbf{M}^{\psi}(\tau)]^{1/2}$, where $\mathbf{M}^{\psi}(\tau)$ is the $q \times q$ diagonal matrix with the q largest eigenvalues of $\mathbf{\Gamma}_n^{\psi}(\tau)$ and $\mathbf{V}_n^{\psi}(\tau)$ the $n \times q$ matrix of the corresponding normalized eigenvectors. Since, by definition, $\widehat{\mathbf{R}}_n(t/T) := \widehat{\mathbf{V}}_{n,T}^{\widehat{Z}}(t/T)[\widehat{\mathbf{M}}_{n,T}^{\widehat{Z}}(t/T)]^{1/2}$, from (A.87) and (A.88), it follows that

$$\max_{t \in \mathcal{T}_T} \left\| \boldsymbol{\ell}_i' \left(\widehat{\mathbf{R}}_{n,T}(t/T) - \mathbf{R}_n(t/T)\mathbf{S}(t/T) \right) \right\| = O_{\mathrm{P}}(\vartheta_{n,T,r^*}^{1/2} \log^{1/\varphi} T) \tag{A.89}$$

as $n, T \to \infty$, uniformly in i.

Finally, recall the definitions

$$\mathbf{C}_n(t/T; L) := [\mathbf{A}_n(t/T; L)]^{-1}\mathbf{R}_n(t/T) \quad \text{and} \quad \widehat{\mathbf{C}}_{n,T}(t; L) := [\widehat{\mathbf{A}}_{n,T}(t/T; L)]^{-1}\widehat{\mathbf{R}}_{n,T}(t/T)$$

of the impulse response functions and their estimators. Since $[\widehat{\mathbf{A}}_{n,T}(t/T; L)]$ is block-diagonal, so is $[\widehat{\mathbf{A}}_{n,T}(t/T; L)]^{-1}$. For any given $1 \leq i \leq n$, label as k_i the block containing the ith component x_{it} of $\mathbf{x}_{nt}$, with $1 \leq k_i \leq m$ and let $\mathcal{I}_i := \{h \in \mathbb{N}_0 : (k_i - 1)(q+1) + 1 \leq h \leq k_i(q+1)\}$ denote the set of indexes corresponding to those components of $\mathbf{x}_{nt}$ belonging to block k_i. Letting $\boldsymbol{m}_j$ stand for the jth vector in the q-dimensional canonical basis (the vector with 1 in entry j and 0 elsewhere), the (i,j) entry of $\widehat{\mathbf{C}}_{n,T}(t; L)$ is

$$\widehat{c}_{ij;n,T}(t; L) = \sum_{h \in \mathcal{I}_i} \boldsymbol{\ell}_i' [\widehat{\mathbf{A}}_{n,T}(t/T; L)]^{-1} \boldsymbol{\ell}_h \boldsymbol{\ell}_h' \widehat{\mathbf{R}}_{n,T}(t/T)\boldsymbol{m}_j;$$

note that the sum is only over $(q+1)$ elements, which is finite for any $n \in \mathbb{N}_0$. Therefore, we can use (A.67) and (A.89) to show that, for any given lag $k \geq 0$,

$$\max_{t \in \mathcal{T}_T} |\widehat{c}_{ijk;n,T}(t) - s_j(t)c_{ijk}(t/T)| = O_{\mathrm{P}}(\vartheta_{n,T,r^*}^{1/2} \log^{1/\varphi} T), \tag{A.90}$$

as $n, T \to \infty$, uniformly in i and j. Moreover, from Assumption (B3), for any given $k \geq 0$, as $T \to \infty$,

$$\sup_{i \in \mathbb{N}_0} \max_{1 \leq j \leq q} \max_{t \in \mathcal{T}_T} \left| c_{ijk}^*(t) - c_{ijk}(t/T) \right| \leq C_\chi \rho_\chi^k / T = O(T^{-1}). \tag{A.91}$$

Combining (A.90) and (A.91) completes the proof. □

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