Generalized Dynamic Factor Models and Volatilities: Consistency, Rates, and Prediction Intervals

Online Appendix

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January 4, 2020

Contents

A Technical Appendix

A.1 Proofs of Lemmas 1, 2, and 3

PROOF OF LEMMA 1. From Assumption L1*(v)*, for any $i = 1, \dots, n$ and any $\theta \in [-\pi, \pi]$,

$$|d_i(e^{-i\theta})| \leq \sum_{k=0}^{\infty} |d_{ik} e^{-i\theta k}| \leq \sum_{k=0}^{\infty} |d_{ik}| \leq M_2.$$

Let $\sigma_{ij}^Z(\theta)$ stand for entry (i,j) of $\mathbf{\Sigma}_n^Z(\theta)$. From Assumption L1*(iv)*, for all $n > \bar{n}$, we have

$$\begin{aligned}\sup_{\theta\in[-\pi,\pi]} \lambda_{n1}^Z(\theta) &\leq \sup_{\theta\in[-\pi,\pi]} \max_{i=1,\dots,n} \sum_{j=1}^{n} |\sigma_{ij}^Z(\theta)| \\ &= \sup_{\theta\in[-\pi,\pi]} \max_{i=1,\dots,n} \frac{1}{2\pi} \sum_{j=1}^{n} |d_i(e^{-i\theta})\text{Cov}(v_{it}, v_{jt}) d_j(e^{i\theta})| \leq M_2^2 C^v / 2\pi.\end{aligned}$$

where we used the fact that $\lambda_{n1}^Z(\theta) = \|\mathbf{\Sigma}_n^Z(\theta)\| \leq \|\mathbf{\Sigma}_n^Z(\theta)\|_1 = \max_{i=1,\dots,n} \sum_{j=1}^n |\sigma_{ij}^Z(\theta)|$. This proves part *(i)*. Parts *(ii)* and *(iii)* are consequences of Assumption L3, part *(i)* above, and Weyl's inequality (Weyl, 1912). □

PROOF OF LEMMA 2. Part *(i)* follows from Proposition 4 in Forni et al. (2017). Parts *(ii)* and *(iii)* are consequences of Assumption L5, part *(i)* above, and Weyl's inequality. □

PROOF OF LEMMA 3. Parts *(i)*-*(iii)* follow as in Lemma 1, parts *(iv)*-*(vi)* as in Lemma 2. □

A.2 Estimation of spectral densities

LEMMA A1. *Let $\sigma_{ij}^Y(\theta)$ and $\widehat{\sigma}_{ij}^Y(\theta)$ stand for the (i,j) entries of $\mathbf{\Sigma}_n^Y(\theta)$ and $\widehat{\mathbf{\Sigma}}_n^Y(\theta)$, respectively. Then,*
(i) letting $\theta_h := \pi h / B_T$ with $|h| \leq B_T$, under Assumptions (L1)-(L2),

$$\text{E}\left[\max_{|h|\leq B_T} \left|\widehat{\sigma}_{ij}^Y(\theta_h) - \sigma_{ij}^Y(\theta_h)\right|^2\right] \leq \frac{C_1 B_T^2}{T} + \frac{C_2}{B_T^2},$$

where $C_1 > 0$ and $C_2 > 0$ are finite and independent of i and j;
(i) letting $\theta_\ell := \pi \ell / M_T$ with $|\ell| \leq M_T$, under Assumptions (V1)-(V2),

$$\text{E}\left[\max_{|\ell|\leq M_T} \left|\widehat{\sigma}_{ij}^h(\theta_\ell) - \sigma_{ij}^h(\theta_\ell)\right|^2\right] \leq \frac{C_3 M_T^2}{T} + \frac{C_4}{M_T^2},$$

where $C_3 > 0$ and $C_4 > 0$ are finite and independent of i and j.

PROOF OF LEMMA A1. For any given i, j, we have

$$\begin{aligned}\text{E}\left[\max_{|h|\leq B_T} \left|\widehat{\sigma}_{ij}^Y(\theta_h) - \sigma_{ij}^Y(\theta_h)\right|^2\right] &= \text{E}\left[\max_{|h|\leq B_T} \left|\widehat{\sigma}_{ij}^Y(\theta_h) - \text{E}[\widehat{\sigma}_{ij}^Y(\theta_h)] + \text{E}[\widehat{\sigma}_{ij}^Y(\theta_h)] - \sigma_{ij}^Y(\theta_h)\right|^2\right] \\ &\leq 2\text{E}\left[\max_{|h|\leq B_T} \left|\widehat{\sigma}_{ij}^Y(\theta_h) - \text{E}[\widehat{\sigma}_{ij}^Y(\theta_h)]\right|^2\right] + 2\text{E}\left[\max_{|h|\leq B_T} \left|\text{E}[\widehat{\sigma}_{ij}^Y(\theta_h)] - \sigma_{ij}^Y(\theta_h)\right|^2\right] = 2(I + II), \text{ say.}\end{aligned} \tag{A1}$$

Considering the first term I, under Assumptions (L1*ii*), (L1*v*), (L1*vii*), and (L1*viii*) (finite fourth-order innovation moments and summability of common and idiosyncratic coefficients), we have that the variance of the lag-window estimator is such that

$$\max_{|h|\leq B_T} \text{E}\left[\left|\widehat{\sigma}_{ij}^Y(\theta_h) - \text{E}[\widehat{\sigma}_{ij}^Y(\theta_h)]\right|^2\right] \leq C_1^* B_T / T, \tag{A2}$$

for some finite $C_1^* > 0$ independent of i and j (see also the first term on the right-hand side of equation (5) in Hallin and Liška, 2007). This is a classical result which is proved, for example, in Theorem 5A of Parzen (1957). Then,

$$\begin{aligned}I = \text{E}\left[\max_{|h|\leq B_T} \left|\widehat{\sigma}_{ij}^Y(\theta_h) - \text{E}[\widehat{\sigma}_{ij}^Y(\theta_h)]\right|^2\right] &\leq \sum_{|h|\leq B_T} \text{E}\left[\left|\widehat{\sigma}_{ij}^Y(\theta_h) - \text{E}[\widehat{\sigma}_{ij}^Y(\theta_h)]\right|^2\right] \\ &\leq (2B_T + 1) \max_{|h|\leq B_T} \text{E}\left[\left|\widehat{\sigma}_{ij}^Y(\theta_h) - \text{E}[\widehat{\sigma}_{ij}^Y(\theta_h)]\right|^2\right] \leq C_1 B_T^2 / T,\end{aligned} \tag{A3}$$

where $C_1 > 0$ is finite and independent of i and j (see also Chapter 6 by Priestley, 2001).

Turning to II, (see also Proposition 6 in Forni et al., 2017)

$$
\begin{aligned}
2\pi\big|\mathrm{E}[\widehat{\sigma}^Y_{ij}(\theta_h)] - \sigma^Y_{ij}(\theta_h)\big| &= \Bigg|\sum_{k=-T+1}^{T-1} \mathrm{K}\left(k/B_T\right)\mathrm{E}[\widehat{\gamma}^Y_{ijk}]e^{-ik\theta_h} - \sum_{k=-\infty}^{\infty}\gamma^Y_{ijk}e^{-ik\theta_h}\Bigg| \\
&\le \Bigg|\sum_{k=-T+1}^{T-1}\left(\mathrm{K}\left(k/B_T\right)-1\right)\gamma^Y_{ijk}e^{-ik\theta_h}\Bigg| + \Bigg|\sum_{k=-T+1}^{T-1}\mathrm{K}\left(k/B_T\right)\frac{|k|}{T}\gamma^Y_{ijk}e^{-ik\theta_h}\Bigg| + \Bigg|\sum_{|k|\ge T}\gamma^Y_{ijk}e^{-ik\theta_h}\Bigg| \\
&= III + IV + V, \text{ say,}
\end{aligned}
\tag{A4}
$$

owing to the fact that $\mathrm{E}[\widehat{\gamma}^Y_{ijk}] = \gamma^Y_{ijk}\left(1-\frac{|k|}{T}\right)$. In order to bound each term of (A4), note that, because of Assumption (L2), there exists a finite constant $D > 0$ and a constant $\phi \in (0, 1)$, both independent of i and j, such that

$$
|\gamma^Y_{ijk}| \le |\gamma^X_{ijk}| + |\gamma^Z_{ijk}| \le D\phi^{|k|}. \tag{A5}
$$

For term III in (A4), using (A5) and the Bartlett kernel, $\mathrm{K}\left(k/B_T\right) = (1 - |k|/B_T)$, we have

$$
III \le D\sum_{k=-\infty}^{\infty}\phi^{|k|}\frac{|k|}{B_T} \le 2D\phi/(1-\phi^2)B_T, \tag{A6}
$$

irrespective of i, j, and θ_h. Similarly, for terms IV and V,

$$
IV \le D\sum_{k=-\infty}^{\infty}\phi^{|k|}|k|/T \le 2D\phi/(1-\phi^2)T \quad \text{and} \quad V \le D\sum_{|k|\ge T}\phi^{|k|}|k|/T \le 2D\phi/(1-\phi^2)T, \tag{A7}
$$

irrespective of i, j, and θ_h, and since $|k|/T > 1$ when $|k| \ge T$. By substituting (A6) and (A7) into (A4), we obtain that $II \le (C_2/B_T^2)$ with $C_2 > 0$ finite and independent of i and j. This proves part *(i)*. Part *(ii)* follows along the same lines. □

A.3 Proof of Proposition 1

From Lemma A1*(i)*, we have the following (see also Lemma 1 in Forni et al., 2017)

$$
\begin{aligned}
&\mathrm{E}\left[\max_{|h|\le B_T}\frac{1}{n^2}\Big\|\widehat{\boldsymbol{\Sigma}}^Y_n(\theta_h) - \boldsymbol{\Sigma}^Y_n(\theta_h)\Big\|^2\right] \le \mathrm{E}\left[\max_{|h|\le B_T}\frac{1}{n^2}\mathrm{tr}\Big\{\Big(\widehat{\boldsymbol{\Sigma}}^Y_n(\theta_h) - \boldsymbol{\Sigma}^Y_n(\theta_h)\Big)\Big(\widehat{\boldsymbol{\Sigma}}^Y_n(\theta_h) - \boldsymbol{\Sigma}^Y_n(\theta_h)\Big)\Big\}\right] \\
&= \frac{1}{n^2}\mathrm{E}\left[\max_{|h|\le B_T}\sum_{i=1}^n\sum_{j=1}^n\big|\widehat{\sigma}^Y_{ij}(\theta_h) - \sigma^Y_{ij}(\theta_h)\big|^2\right] \le \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n\mathrm{E}\left[\max_{|h|\le B_T}\big|\widehat{\sigma}^Y_{ij}(\theta_h) - \sigma^Y_{ij}(\theta_h)\big|^2\right] \le C_1B_T^2/T + C_2/B_T^2.
\end{aligned}
\tag{A8}
$$

Therefore, by Chebychev's inequality and (A8)

$$
\max_{|h|\le B_T}\frac{1}{n}\Big\|\widehat{\boldsymbol{\Sigma}}^Y_n(\theta_h) - \boldsymbol{\Sigma}^Y_n(\theta_h)\Big\| = O_{\mathrm{P}}\left(\max\left(B_T/\sqrt{T}, 1/B_T\right)\right). \tag{A9}
$$

Let $\boldsymbol{\ell}_i$ denote the n-dimensional vector with 1 in entry i and 0 elsewhere. Then,

$$
\begin{aligned}
&\mathrm{E}\left[\max_{|h|\le B_T}\frac{1}{n}\Big\|\boldsymbol{\ell}_i'\Big(\widehat{\boldsymbol{\Sigma}}^Y_n(\theta_h) - \boldsymbol{\Sigma}^Y_n(\theta_h)\Big)\Big\|^2\right] = \mathrm{E}\left[\max_{|h|\le B_T}\frac{1}{n}\boldsymbol{\ell}_i'\Big(\widehat{\boldsymbol{\Sigma}}^Y_n(\theta_h) - \boldsymbol{\Sigma}^Y_n(\theta_h)\Big)\Big(\widehat{\boldsymbol{\Sigma}}^Y_n(\theta_h) - \boldsymbol{\Sigma}^Y_n(\theta_h)\Big)'\boldsymbol{\ell}_i\right] \\
&= \mathrm{E}\left[\max_{|h|\le B_T}\frac{1}{n}\boldsymbol{\ell}_i'\Big(\widehat{\boldsymbol{\Sigma}}^Y_n(\theta_h) - \boldsymbol{\Sigma}^Y_n(\theta_h)\Big)\Big(\widehat{\boldsymbol{\Sigma}}^Y_n(\theta_h) - \boldsymbol{\Sigma}^Y_n(\theta_h)\Big)\boldsymbol{\ell}_i\right] = \frac{1}{n}\mathrm{E}\left[\max_{|h|\le B_T}\sum_{j=1}^n\big|\widehat{\sigma}^Y_{ij}(\theta_h) - \sigma^Y_{ij}(\theta_h)\big|^2\right] \\
&\le \frac{1}{n}\sum_{j=1}^n\mathrm{E}\left[\max_{|h|\le B_T}\big|\widehat{\sigma}^Y_{ij}(\theta_h) - \sigma^Y_{ij}(\theta_h)\big|^2\right] \le C_1B_T^2/T + C_2/B_T^2.
\end{aligned}
\tag{A10}
$$

Hence, since C_1 and C_2 in (A10) do not depend on i, by Chebychev's inequality, for any $\epsilon > 0$, there exists $\eta(\epsilon)$ and an integer $T^* = T(\epsilon)$ both independent of i and such that

$$
\mathrm{P}\left(\max_{|h|\le B_T}\frac{1}{(B_T/\sqrt{T} + 1/B_T)\sqrt{n}}\Big\|\boldsymbol{\ell}_i'\Big(\widehat{\boldsymbol{\Sigma}}^Y_n(\theta_h) - \boldsymbol{\Sigma}^Y_n(\theta_h)\Big)\Big\| \ge \eta(\epsilon)\right) < \epsilon,
$$

for all $n \in \mathbb{N}$, $i = 1, \ldots, n$, and $T \geq T^*$, which implies that

$$\max_{i=1,\ldots,n} \max_{|h|\leq B_T} \frac{1}{\sqrt{n}} \left\| \boldsymbol{\ell}_i' \left(\widehat{\boldsymbol{\Sigma}}_n^Y(\theta_h) - \boldsymbol{\Sigma}_n^Y(\theta_h) \right) \right\| = O_{\mathrm{P}} \left(\max \left(B_T/\sqrt{T}, 1/B_T \right) \right). \tag{A11}$$

Note also that (A9) and (A11) hold independently of n. Moreover, using (A9) and Lemma 1*(i)*, we have

$$\begin{aligned}\max_{|h|\leq B_T} \frac{1}{n} \left\| \widehat{\boldsymbol{\Sigma}}_n^Y(\theta_h) - \boldsymbol{\Sigma}_n^X(\theta_h) \right\| &\leq \max_{|h|\leq B_T} \frac{1}{n} \left\| \widehat{\boldsymbol{\Sigma}}_n^Y(\theta_h) - \boldsymbol{\Sigma}_n^Y(\theta_h) \right\| + \max_{|h|\leq B_T} \frac{1}{n} \left\| \boldsymbol{\Sigma}_n^Z(\theta_h) \right\| \\ &= O_{\mathrm{P}} \left(\max \left(B_T/\sqrt{T}, 1/B_T, 1/n \right) \right). \end{aligned} \tag{A12}$$

Then, note that, because of Lemma 1*(i)* and since $\|\boldsymbol{\ell}_i\| = 1$,

$$\max_{i=1,\ldots,n} \max_{|h|\leq B_T} \frac{1}{n} \left\| \boldsymbol{\ell}_i' \boldsymbol{\Sigma}_n^Z(\theta_h) \right\|^2 \leq \max_{\boldsymbol{w}:\|\boldsymbol{w}\|=1} \max_{|h|\leq B_T} \frac{1}{n} \boldsymbol{w}' \boldsymbol{\Sigma}_n^Z(\theta_h) \boldsymbol{\Sigma}_n^Z(\theta_h) \boldsymbol{w} = \frac{1}{n} \left\| \boldsymbol{\Sigma}_n^Z(\theta_h) \right\|^2 = O(1/n). \tag{A13}$$

Hence, from (A10) and (A13), following the same approach as in (A12), it follows that

$$\max_{i=1,\ldots,n} \max_{|h|\leq B_T} \frac{1}{\sqrt{n}} \left\| \boldsymbol{\ell}_i' \left(\widehat{\boldsymbol{\Sigma}}_n^Y(\theta_h) - \boldsymbol{\Sigma}_n^X(\theta_h) \right) \right\| = O_{\mathrm{P}} \left(\max \left(B_T/\sqrt{T}, 1/B_T, 1/\sqrt{n} \right) \right). \tag{A14}$$

It follows from (A12) that, for all $j = 1, \ldots, q$ (see also Lemma 2*(i)* in Forni et al., 2017)

$$\max_{|h|\leq B_T} \frac{1}{n} \left| \widehat{\lambda}_{nj}^Y(\theta_h) - \lambda_{nj}^X(\theta_h) \right| \leq \max_{|h|\leq B_T} \frac{1}{n} \left\| \widehat{\boldsymbol{\Sigma}}_n^Y(\theta_h) - \boldsymbol{\Sigma}_n^X(\theta_h) \right\| = O_{\mathrm{P}} \left(\max \left(B_T/\sqrt{T}, 1/B_T, 1/n \right) \right). \tag{A15}$$

Let $\widehat{\boldsymbol{\Lambda}}_n^Y(\theta_h)$ and $\boldsymbol{\Lambda}_n^X(\theta_h)$ be the $q \times q$ diagonal matrices with the q largest eigenvalues of $\widehat{\boldsymbol{\Sigma}}_n^Y(\theta_h)$ and $\boldsymbol{\Sigma}_n^X(\theta_h)$, respectively. Then, from (A15),

$$\max_{|h|\leq B_T} \frac{1}{n} \left\| \widehat{\boldsymbol{\Lambda}}_n^Y(\theta_h) - \boldsymbol{\Lambda}_n^X(\theta_h) \right\| \leq \frac{1}{n} \sum_{j=1}^q \max_{|h|\leq B_T} \left| \widehat{\lambda}_{nj}^Y(\theta_h) - \lambda_{nj}^X(\theta_h) \right| = O_{\mathrm{P}} \left(\max \left(B_T/\sqrt{T}, 1/B_T, 1/n \right) \right), \tag{A16}$$

and, from Assumption (L3) and Lemma 1*(ii)* (see also Lemma 2*(ii)* in Forni et al., 2017),

$$\max_{|h|\leq B_T} n \left\| (\boldsymbol{\Lambda}_n^X(\theta_h))^{-1} \right\| = O(1), \qquad \max_{|h|\leq B_T} n \left\| (\widehat{\boldsymbol{\Lambda}}_n^Y(\theta_h))^{-1} \right\| = O_{\mathrm{P}}(1). \tag{A17}$$

Moreover, using (A12) and following Lemma 3 in Forni et al. (2017), it can be shown that there exist $q \times q$ complex diagonal matrices $\boldsymbol{\mathcal{J}}(\theta_h)$ with entries having unit modulus, such that

$$\max_{|h|\leq B_T} \left\| \widehat{\mathbf{P}}_n^{Y\dagger}(\theta_h) \mathbf{P}_n^X(\theta_h) - \boldsymbol{\mathcal{J}}(\theta_h) \right\| = O_{\mathrm{P}} \left(\max \left(B_T/\sqrt{T}, 1/B_T, 1/n \right) \right). \tag{A18}$$

Now, note that, because of Assumption (L1*i*), (L1*ii*), (L1*iii*), and (L1*v*), $\sigma_{ii}^X(\theta_h) = \sum_{j=1}^q \lambda_{nj}^X(\theta_h) \, |p_{ij}^X(\theta_h)|^2 \leq M$, for some $M > 0$ finite and independent of i and θ_h. Therefore, from Assumption (L3) we get (see also equation (B5) in Forni et al., 2017)

$$\max_{i=1,\ldots,n} \max_{|h|\leq B_T} \sqrt{n} \left\| \boldsymbol{\ell}_i' \mathbf{P}_n^X(\theta_h) \right\| \leq M^*, \tag{A19}$$

for some $M^* > 0$ finite and independent of n. Therefore, from (A18), (A19) and using (A14), it is possible to prove that (see the proof of Lemma 4 in Forni et al., 2017 for details)

$$\max_{i=1,\ldots,n} \max_{|h|\leq B_T} \sqrt{n} \left\| \boldsymbol{\ell}_i' \left(\mathbf{P}_n^X(\theta_h) \boldsymbol{\mathcal{J}}(\theta_h) - \widehat{\mathbf{P}}_n^Y(\theta_h) \right) \right\| = O_{\mathrm{P}} \left(\max \left(B_T/\sqrt{T}, 1/B_T, 1/\sqrt{n} \right) \right), \tag{A20}$$

and, from (A16), (A17), and (A20), we have that (see the proof of Lemma 4 in Forni et al., 2017 for details)

$$\max_{i=1,\ldots,n} \max_{|h|\leq B_T} \left\| \boldsymbol{\ell}_i' \left(\mathbf{P}_n^X(\theta_h) (\boldsymbol{\Lambda}_n^X(\theta_h))^{1/2} \boldsymbol{\mathcal{J}}(\theta_h) - \widehat{\mathbf{P}}_n^Y(\theta_h) (\widehat{\boldsymbol{\Lambda}}_n^Y(\theta_h))^{1/2} \right) \right\| = O_{\mathrm{P}} \left(\max \left(B_T/\sqrt{T}, 1/B_T, 1/\sqrt{n} \right) \right). \tag{A21}$$

The estimator of the spectral density matrix of $\mathbf{X}_n$ is defined as $\widehat{\boldsymbol{\Sigma}}_n^X(\theta_h) := \widehat{\mathbf{P}}_n^Y(\theta_h) \widehat{\boldsymbol{\Lambda}}_n^Y(\theta_h) \widehat{\mathbf{P}}_n^{Y\dagger}(\theta_h)$, with entries $\widehat{\sigma}_{ij}^X(\theta_h)$. Then, (A21) implies (see also Proposition 7 in Forni et al., 2017)

$$\max_{i,j=1,\ldots,n} \max_{|h|\leq B_T} \left| \widehat{\sigma}_{ij}^X(\theta_h) - \sigma_{ij}^X(\theta_h) \right| = \max_{i,j=1,\ldots,n} \max_{|h|\leq B_T} \left\| \boldsymbol{\ell}_i' (\widehat{\boldsymbol{\Sigma}}_n^X(\theta_h) - \boldsymbol{\Sigma}_n^X(\theta_h)) \boldsymbol{\ell}_j \right\| = O_{\mathrm{P}}(\rho_{nT}), \tag{A22}$$

where $\rho_{nT} := \max\left(B_T/\sqrt{T}, 1/B_T, 1/\sqrt{n}\right)$.

Moving to the autocovariances of the common component, the (i,j) entry γ^X_{ijk} of $\mathbf{\Gamma}^X_{nk}$ is obtained as the inverse Fourier transform

$$\gamma^X_{ijk} = \int_{-\pi}^{\pi} e^{ik\theta}\sigma^X_{ij}(\theta)\mathrm{d}\theta.$$

Denoting by $\widehat{\gamma}^X_{ijk}$ the entries of the estimated autocovariances $\widehat{\mathbf{\Gamma}}^X_{nk}$, we have (see also Proposition 8 in Forni et al., 2017)

$$\begin{aligned}|\widehat{\gamma}^X_{ijk} - \gamma^X_{ijk}| &\le \frac{\pi}{B_T}\sum_{|h|\le B_T}|\widehat{\sigma}^X_{ij}(\theta_h) - \sigma^X_{ij}(\theta_h)| + \frac{\pi}{B_T}\sum_{|h|\le B_T}\max_{\theta_{h-1}\le\theta\le\theta_h}|e^{ik\theta_h}\sigma^X_{ij}(\theta_h) - e^{ik\theta}\sigma^X_{ij}(\theta)|\\ &\le \pi\max_{|h|\le B_T}\left|\widehat{\sigma}^X_{ij}(\theta_h) - \sigma^X_{ij}(\theta_h)\right| + \frac{C_k}{B_T} = O_{\mathrm{P}}(\rho_{nT}),\end{aligned} \tag{A23}$$

where we used (A22), the fact that the functions $\theta\mapsto e^{ik\theta}$ and $\theta\mapsto\sigma^X_{ij}(\theta)$ are of bounded variation, and Assumption (K). Moreover, in view of (A22), (A23) holds uniformly in i and j:

$$\max_{i,j=1,\dots,n}|\widehat{\gamma}^X_{ijk} - \gamma^X_{ijk}| = O_{\mathrm{P}}(\rho_{nT}). \tag{A24}$$

Notice, however, that (A23) does not hold uniformly in k, which poses no problem since we always consider $k\le S$, with $S<\infty$ because of Assumption (L4).

Hereafter, for simplicity of notation and without loss of generality, we assume that $n = m(q+1)$ (with $m\in\mathbb{N}$) and all VARs are of order one; namely, $\mathbf{A}^{(\ell)}(L) = (\mathbf{I}_{q+1} - \mathbf{A}^{(\ell)}_1 L)$ for $\ell = 1,\dots,m$. Consider the traditional Yule-Walker estimator $\widehat{\mathbf{A}}^{(\ell)}_1 := \widehat{\mathbf{\Gamma}}^{X^{(\ell)}}_1\left[\widehat{\mathbf{\Gamma}}^{X^{(\ell)}}_0\right]^{-1}$ of $\mathbf{A}^{(\ell)}_1$ (see also (2.8)). Then, the $n\times n$ block-diagonal VAR operator $\mathbf{A}_n(L) = (\mathbf{I}_n - \mathbf{A}_{n1}L)$ with diagonal blocks $\mathbf{I}_{q+1} - \mathbf{A}^{(1)}_1,\dots,\mathbf{I}_{q+1} - \mathbf{A}^{(m)}_1$ has block-diagonal estimator $\widehat{\mathbf{A}}_n(L) = (\mathbf{I}_n - \widehat{\mathbf{A}}_{n1}L)$, with diagonal blocks $\mathbf{I}_{q+1} - \widehat{\mathbf{A}}^{(1)}_1,\dots,\mathbf{I}_{q+1} - \widehat{\mathbf{A}}^{(m)}_1$. As a consequence of (A23), we have (see also Proposition 9 Forni et al., 2017)

$$\max_{\ell=1,\dots,m}\|\widehat{\mathbf{A}}^{(\ell)}_1 - \mathbf{A}^{(\ell)}_1\| = O_{\mathrm{P}}(\rho_{nT}). \tag{A25}$$

Let $\mathbf{a}'_i$ and $\widehat{\mathbf{a}}'_i$ denote the i-th rows of $\mathbf{A}_{n1}$ and $\widehat{\mathbf{A}}_{n1}$, respectively. Since $\mathbf{A}_{n1}$ has only $n(q+1)^2$ non-zero entries, and since each of its n rows has only $(q+1)$ non-zero entries, we also have

$$\max_{i=1,\dots,n}\|\widehat{\mathbf{a}}'_i - \mathbf{a}'_i\| = O_{\mathrm{P}}(\rho_{nT}) \quad\text{and}\quad \frac{1}{\sqrt{n}}\|\widehat{\mathbf{A}}_{n1} - \mathbf{A}_{n1}\| = O_{\mathrm{P}}(\rho_{nT}), \tag{A26}$$

where uniformity over i is a consequence of (A24).

Turning to $\mathbf{H}_n$ and $\widehat{\mathbf{H}}_n$, with i-th rows $\boldsymbol{h}'_i$ and $\widehat{\boldsymbol{h}}'_i$, respectively, we have

$$\max_{i=1,\dots,n}\|\widehat{\boldsymbol{h}}'_i - \boldsymbol{h}'_i\mathbf{J}\| = O_{\mathrm{P}}(\rho_{nT}) \quad\text{and}\quad \frac{1}{\sqrt{n}}\|\widehat{\mathbf{H}}_n - \mathbf{H}_n\mathbf{J}\| = O_{\mathrm{P}}(\rho_{nT}), \tag{A27}$$

where $\mathbf{J}$ is some $q\times q$ diagonal matrix with entries ±1 (see also Proposition 10 in Forni et al., 2017 which since $\widehat{\mathbf{H}}_n$ is a matrix of eigenvectors is based on steps similar to those leading to (A20)). Because $\mathbf{A}_n(L)$ and $\widehat{\mathbf{A}}_n(L)$ are block-diagonal, so are $\mathbf{B}_n(L) = [\mathbf{A}_n(L)]^{-1}$ and $\widehat{\mathbf{B}}_n(L) = [\widehat{\mathbf{A}}_n(L)]^{-1}$. Hence, all rows of $\mathbf{B}_n(L)$ and $\widehat{\mathbf{B}}_n(L)$, irrespective of n, have at most $(q+1)$ non-zero entries. It thus follows from (A26) and (A27) that, for any $k\ge0$,

$$\max_{i=1,\dots,n}\|\widehat{\mathbf{b}}'_{ik} - \mathbf{b}'_{ik}\mathbf{J}\| = O_{\mathrm{P}}(\rho_{nT}). \tag{A28}$$

This completes the proof of part *(a)* of the proposition.

The estimator $\widehat{\mathbf{A}}_n(L)$ provides an estimator $\widehat{\mathbf{Y}}^*_n := \widehat{\mathbf{A}}_n(L)\mathbf{Y}_n$ for the filtered process $\mathbf{Y}^*_n$. Consider the estimated factors

$$\begin{aligned}\widehat{\mathbf{u}}_t = \frac{1}{n}\widehat{\mathbf{H}}'_n\widehat{\mathbf{Y}}^*_{nt} &= \frac{1}{n}\big(\widehat{\mathbf{H}}'_n\widehat{\mathbf{A}}_n(L) - \mathbf{J}\mathbf{H}'_n\mathbf{A}_n(L)\big)\mathbf{Y}_{nt} + \frac{1}{n}\mathbf{J}\mathbf{H}'_n\mathbf{A}_n(L)\mathbf{Y}_{nt}\\ &= \frac{1}{n}\big(\widehat{\mathbf{H}}'_n\widehat{\mathbf{A}}_n(L) - \mathbf{J}\mathbf{H}'_n\mathbf{A}_n(L)\big)\mathbf{Y}_{nt} + \frac{1}{n}\mathbf{J}\mathbf{H}'_n\mathbf{A}_n(L)\mathbf{X}_{nt} + \frac{1}{n}\mathbf{J}\mathbf{H}'_n\mathbf{A}_n(L)\mathbf{Z}_{nt}\\ &= \frac{1}{n}\big(\widehat{\mathbf{H}}'_n\widehat{\mathbf{A}}_n(L) - \mathbf{J}\mathbf{H}'_n\mathbf{A}_n(L)\big)\mathbf{Y}_{nt} + \frac{1}{n}\mathbf{J}\mathbf{H}'_n\mathbf{H}_n\mathbf{u}_t + \frac{1}{n}\mathbf{J}\mathbf{H}'_n\mathbf{A}_n(L)\mathbf{Z}_{nt}\\ &= \frac{1}{n}\big(\widehat{\mathbf{H}}'_n\widehat{\mathbf{A}}_n(L) - \mathbf{J}\mathbf{H}'_n\mathbf{A}_n(L)\big)\mathbf{Y}_{nt} + \mathbf{J}\mathbf{u}_t + \frac{1}{n}\mathbf{J}\mathbf{H}'_n\mathbf{A}_n(L)\mathbf{Z}_{nt},\end{aligned}$$

where we used the identification constraints in Assumption (Ii). Then,

$$\max_{t=1,\ldots,T}\|\widehat{\mathbf{u}}_t-\mathbf{J}\mathbf{u}_t\|\le \max_{t=1,\ldots,T}\frac{1}{n}\big\|\big[\widehat{\mathbf{H}}_n'\widehat{\mathbf{A}}_n(L)-\mathbf{J}\mathbf{H}_n'\mathbf{A}_n(L)\big]\mathbf{Y}_{nt}\big\|+\max_{t=1,\ldots,T}\frac{1}{n}\big\|\mathbf{J}\mathbf{H}_n'\mathbf{A}_n(L)\mathbf{Z}_{nt}\big\|$$
$$=A+B,\ \text{say.} \tag{A29}$$

Term A in (A29) is such that

$$\max_{t=1,\ldots,T}\frac{1}{n}\big\|\big(\widehat{\mathbf{H}}_n'\widehat{\mathbf{A}}_n(L)-\mathbf{J}\mathbf{H}_n'\mathbf{A}_n(L)\big)\mathbf{Y}_{nt}\big\| \tag{A30}$$
$$\le\frac{1}{\sqrt{n}}\Big[\big\|\widehat{\mathbf{H}}_n-\mathbf{H}_n\mathbf{J}\big\|+\big\|\widehat{\mathbf{H}}_n'\widehat{\mathbf{A}}_{n1}-\mathbf{J}\mathbf{H}_n'\mathbf{A}_{n1}\big\|\Big]\max_{t=1,\ldots,T}\big\|\mathbf{Y}_{nt}/\sqrt{n}\big\| \tag{A31}$$
$$=O_{\mathrm{P}}(\rho_{nT})\max_{t=1,\ldots,T}\big\|\mathbf{Y}_{nt}/\sqrt{n}\big\|,$$

because of (A26) and (A27). Moreover,

$$\max_{t=1,\ldots,T}\big\|\mathbf{Y}_{nt}/\sqrt{n}\big\|\le\max_{t=1,\ldots,T}\big\|\mathbf{X}_{nt}/\sqrt{n}\big\|+\max_{t=1,\ldots,T}\big\|\mathbf{Z}_{nt}/\sqrt{n}\big\|$$
$$\le\frac{1}{\sqrt{n}}\max_{t=1,\ldots,T}\Big\|\sum_{k=0}^{\infty}\mathbf{B}_{nk}\mathbf{u}_{t-k}\Big\|+\max_{t=1,\ldots,T}\max_{i=1,\ldots,n}|Z_{it}|$$
$$\le\max_{t=1,\ldots,T}\max_{i=1,\ldots,n}\sum_{k=0}^{\infty}\|\mathbf{b}_{ik}'\|\,\|\mathbf{u}_t\|+\max_{t=1,\ldots,T}\max_{i=1,\ldots,n}|Z_{it}|$$
$$\le M_1\sqrt{q}\max_{t=1,\ldots,T}\max_{j=1,\ldots,q}|u_{jt}|+\max_{t=1,\ldots,T}\max_{i=1,\ldots,n}|Z_{it}|=AI+AII,\ \text{say.} \tag{A32}$$

In Assumption (Ti) and (Tiii) we can set $K_u=1$ and $K_Z=1$ by replacing u_{jt} and Z_{it} with $u_{jt}/\|u_{jt}\|_{\psi_1}$ and $Z_{it}/\|Z_{it}\|_{\psi_1}$, respectively; since the sub-exponential norms are assumed to be finite, there is no loss of generality in this choice. Now, using Assumption (Ti), and since, by Assumption (L1i), $\mathrm{E}[u_{jt}]=0$ for all j, we have, for all λ such that $|\lambda|\le 1/e$ (see also Lemma 5.15 in Vershynin, 2012),

$$\max_{j=1,\ldots,q}\mathrm{E}[\exp(\lambda u_{jt})]=\max_{j=1,\ldots,q}\left\{1+\lambda\mathrm{E}[u_{jt}]+\sum_{p=2}^{\infty}\frac{\lambda^p\mathrm{E}[(u_{jt})^p]}{p!}\right\}\le\left\{1+\sum_{p=2}^{\infty}\frac{|\lambda|^pp^p}{p!}\right\}$$
$$\le\left\{1+\sum_{p=2}^{\infty}|\lambda|^pe^p\right\}=1+e^2\lambda^2\le\exp(\lambda^2e^2), \tag{A33}$$

where we used the fact that $p!\ge(p/e)^p$. Then, for any $\epsilon>0$ and $|\lambda|\le 1/e$, we have from (A33) that

$$\mathrm{P}(u_{jt}>\epsilon)=\mathrm{P}\left(\exp\left(u_{jt}\lambda\right)>\exp\left(\epsilon\lambda\right)\right)\le\mathrm{E}\left[\exp\left(u_{jt}\lambda\right)\right]\exp\left(-\epsilon\lambda\right)\le\exp\left(\lambda^2e^2-\epsilon\lambda\right). \tag{A34}$$

Similarly, we have $\mathrm{P}(u_{jt}<-\epsilon)\le\exp\left(\lambda^2e^2-\epsilon\lambda\right)$. Without loss of generality, we may set $\lambda=1/3$, which yields $\mathrm{P}(|u_{jt}|>\epsilon)\le K_u^*\exp\left(-\epsilon/3\right)$ for some finite $K_u^*>0$. By Bonferroni inequality, we then obtain

$$\mathrm{P}\Big(\max_{t=1,\ldots,T}\max_{j=1,\ldots,q}|u_{jt}|>\epsilon\Big)\le TK_u^*\exp\left(-\epsilon/3\right). \tag{A35}$$

Therefore, term AI on the right-hand side of (A32) is $O_{\mathrm{P}}(\log T)$. Turning to AII on the right-hand side of (A32), notice that, since $\|\boldsymbol{\ell}_i\|=1$, then

$$\max_{i=1,\ldots,n}\|Z_{it}\|_{\psi_1}=\max_{i=1,\ldots,n}\|\boldsymbol{\ell}_i'\mathbf{Z}_t\|_{\psi_1}\le\sup_{\boldsymbol{w}_n:\|\boldsymbol{w}_n\|=1}\|\boldsymbol{w}_n'\mathbf{Z}_t\|_{\psi_1}\le K_Z,$$

for all $n\in\mathbb{N}$ by Assumption (T1iii). Therefore, using Bonferroni inequality, we obtain

$$\mathrm{P}\Big(\max_{t=1,\ldots,T}\max_{i=1,\ldots,n}|Z_{it}|>\epsilon\Big)\le TK_Z^*\exp\left(-\epsilon/3\right). \tag{A36}$$

Hence, AII on the right-hand side of (A32) is $O_{\mathrm{P}}(\log T)$. By substituting (A32) into (A30), we conclude that term A in (A29) is $O_{\mathrm{P}}(\rho_{nT}\log T)$.

Turning to term B in (A29), we have (note that $\|\mathbf{J}\| = 1$)

$$\begin{aligned}
\max_{t=1,\ldots,T} \frac{1}{n}\left\|\mathbf{J}\mathbf{H}_n'\mathbf{A}_n(L)\mathbf{Z}_{nt}\right\| &\le \max_{t=1,\ldots,T} \frac{1}{n}\left\|\mathbf{H}_n'\mathbf{Z}_{nt}\right\| + \max_{t=1,\ldots,T} \frac{1}{n}\left\|\mathbf{H}_n'\mathbf{A}_{n1}\mathbf{Z}_{nt-1}\right\| \\
&\le \max_{t=1,\ldots,T} \frac{1}{\sqrt{n}}\left\|\mathbf{P}_n^{X^*\prime}\mathbf{Z}_{nt}\right\| + \max_{t=1,\ldots,T} \frac{1}{\sqrt{n}}\left\|\mathbf{P}_n^{X^*\prime}\mathbf{A}_{n1}\mathbf{Z}_{nt-1}\right\| \\
&\le \sqrt{\frac{q}{n}}\Big(\max_{t=1,\ldots,T}\max_{j=1,\ldots,q} \left|\mathbf{p}_{nj}^{X^*\prime}\mathbf{Z}_{nt}\right| + \max_{t=1,\ldots,T}\max_{j=1,\ldots,q} \left|\mathbf{p}_{nj}^{X^*\prime}\mathbf{A}_{n1}\mathbf{Z}_{nt-1}\right|\Big) \\
&= BI \ + BII, \text{ say,}
\end{aligned} \tag{A37}$$

where we used the identification constraints of Assumption (I*i*). Repeating the same arguments as above, we can show that, if $|\lambda| \le 1/e$, then

$$\sup_{\boldsymbol{w}_n:\|\boldsymbol{w}_n\|=1} \mathrm{E}[\exp(\lambda \boldsymbol{w}_n'\mathbf{Z}_{nt})] \le \exp(\lambda^2 e^2). \tag{A38}$$

Without loss of generality we can set $\lambda = 1/3$ in (A38), and, since $\|\mathbf{p}_{nj}^{X^*}\| = 1$, using the same reasoning as in (A34) and the Bonferroni inequality, for any $\epsilon > 0$, we obtain, for some finite $K_Z^* > 0$,

$$\mathrm{P}\Big(\max_{t=1,\ldots,T}\max_{j=1,\ldots,q} \left|\mathbf{p}_{nj}^{X^*\prime}\mathbf{Z}_{nt}\right| > \epsilon) \le K_Z^* T q \exp\left(-\epsilon/3\right).$$

Therefore, BI on the right-hand side of (A37) is such that

$$BI = \sqrt{q/n} \max_{j=1,\ldots,q}\max_{t=1,\ldots,T} \left|\mathbf{p}_{nj}^{X^*\prime}\mathbf{Z}_{nt}\right| = O_{\mathrm{P}}\left(\log T/\sqrt{n}\right). \tag{A39}$$

Last, because of stationarity in Assumption (L4*iii*), $\|\mathbf{A}_{n1}\| < 1$ thus $\|\mathbf{p}_{nj}^{X^*}\mathbf{A}_{n1}\| \le 1$, and, since if Assumption (T*iii*) holds for $\sup_{\boldsymbol{w}_n:\|\boldsymbol{w}_n\|=1}$ it also holds for $\sup_{\boldsymbol{w}_n:\|\boldsymbol{w}_n\|\le 1}$, the same reasoning as for (A39) yields

$$BII = \sqrt{q/n} \max_{j=1,\ldots,q}\max_{t=1,\ldots,T} \left|\mathbf{p}_{nj}^{X^*\prime}\mathbf{A}_{n1}\mathbf{Z}_{nt-1}\right| = O_{\mathrm{P}}\left(\log T/\sqrt{n}\right). \tag{A40}$$

Substituting (A39) and (A40) in (A37), we conclude that term B is $O_{\mathrm{P}}(\rho_{nT}\log T/\sqrt{n})$. Therefore, term A, which is $O_{\mathrm{P}}(\rho_{nT}\log T)$, dominates in (A29); part *(b)* of the proposition follows.

From parts *(a)* and *(b)* and (A25), it immediately follows that

$$\max_{i=1,\ldots,n}\max_{t=1,\ldots,T} |\widehat{e}_{it} - e_{it}| = \max_{i=1,\ldots,n}\max_{t=1,\ldots,T} |\widehat{\boldsymbol{h}}_i'\widehat{\mathbf{u}}_t - \boldsymbol{h}_i'\mathbf{u}_t| = O_{\mathrm{P}}(\rho_{nT}\log T). \tag{A41}$$

Now, for some finite $M > 0$ that does not depend on i,

$$\max_{i=1,\ldots,n}\left\|\sum_{k=0}^{\bar{k}_1}\widehat{\mathbf{b}}_{ik}' - \sum_{k=0}^{\infty}\mathbf{b}_{ik}'\mathbf{J}\right\| \le \max_{i=1,\ldots,n}\left\|\sum_{k=0}^{\bar{k}_1}(\widehat{\mathbf{b}}_{ik}' - \mathbf{b}_{ik}'\mathbf{J})\right\| + \max_{i=1,\ldots,n}\sum_{|k|\ge\bar{k}_1}\|\mathbf{b}_{ik}'\|\frac{|k|}{\bar{k}_1} \le O_{\mathrm{P}}(\rho_{nT}) + \frac{M}{\bar{k}_1};$$

the bound on the first term on the right-hand side follows from (A28) and Proposition 3.6 in Lütkepohl (2005), which holds for any sequence $\bar{k}_1 \to \infty$, while for the second term we used Assumption (L1*ii*). By taking $\bar{k}_1$ large enough ($\bar{k}_1 \simeq B_T^{-1}$, say), the second term can be made smaller than the first one. Since $X_{it} = \mathbf{b}_i'(L)\mathbf{u}_t$ and $\widehat{X}_{it} := \widehat{\mathbf{b}}_i'(L)\widehat{\mathbf{u}}_t$, it follows that

$$\max_{i=1,\ldots,n}\max_{t=1,\ldots,T} |\widehat{X}_{it} - X_{it}| = O_{\mathrm{P}}(\rho_{nT}\log T)$$

and

$$\max_{i=1,\ldots,n}\max_{t=1,\ldots,T} |\widehat{Z}_{it} - Z_{it}| = \max_{i=1,\ldots,n}\max_{t=1,\ldots,T} |Y_{it} - \widehat{X}_{it} - Y_{it} + X_{it}| = O_{\mathrm{P}}(\rho_{nT}\log T). \tag{A42}$$

For simplicity of notation and without loss of generality as far as this proof is concerned, let us assume that

$$[d_i(L)]^{-1} =: c_i(L) = (1 - c_{i1}L) \quad \text{and} \quad \widehat{c}_i(L) = (1 - \widehat{c}_{i1}L).$$

Then, for any given $i = 1, \ldots, n$, we define the estimator

$$\widehat{c}_{i1} := \sum_{t=2}^{T}\widehat{Z}_{it}\widehat{Z}_{i,t-1}\left(\sum_{t=2}^{T}\widehat{Z}_{i,t-1}^2\right)^{-1}. \tag{A43}$$

For the numerator of (A43), we have

$$\left|\frac{1}{T}\sum_{t=2}^{T}\left(\widehat{Z}_{it}\widehat{Z}_{i,t-1}-Z_{it}Z_{i,t-1}\right)\right| \leq \left|\frac{1}{T}\sum_{t=2}^{T}\left(\widehat{Z}_{it}-Z_{it}\right)Z_{i,t-1}\right| + \left|\frac{1}{T}\sum_{t=2}^{T}\left(\widehat{Z}_{it}-Z_{it}\right)Z_{it}\right| \\ + \left|\frac{1}{T}\sum_{t=2}^{T}\left(\widehat{Z}_{it}-Z_{it}\right)\left(\widehat{Z}_{i,t-1}-Z_{i,t-1}\right)\right| = CI + CII + CIII, \text{ say.} \tag{A44}$$

First consider term CI:

$$CI \leq \max_{t=1,\ldots,T}|\widehat{Z}_{it}-Z_{it}|\left|\frac{1}{T}\sum_{t=2}^{T}Z_{i,t-1}\right| \leq \left(\max_{t=1,\ldots,T}|\widehat{Z}_{it}-Z_{it}|\right)\left(\max_{t=1,\ldots,T}|Z_{i,t-1}|\right) = O_{\mathrm{P}}(\rho_{nT}\log^2 T),$$

uniformly over i because of (A42) and (A36). A similar reasoning shows that CII in (A44) also is $O_{\mathrm{P}}(\rho_{nT}\log^2 T)$ uniformly in i, while $CIII = O_{\mathrm{P}}(\rho_{nT}\log T)$ uniformly over i. Turning to the denominator of (A43), we can show that, uniformly in i,

$$\left|\frac{1}{T}\sum_{t=2}^{T}\left(\widehat{Z}^2_{i,t-1}-Z^2_{it}\right)\right| = O_{\mathrm{P}}(\rho_{nT}\log T). \tag{A45}$$

Consider then the (infeasible) oracle estimator $\widetilde{c}_{i1} := \sum_{t=2}^{T} Z_{it}Z_{i,t-1}/\sum_{t=2}^{T} Z^2_{i,t-1}$ we would construct if the idiosyncratic components were observed. That oracle is such that

$$\widetilde{c}_{i1} - c_{i1} = \sum_{t=2}^{T} Z_{i,t-1}v_{it}\left(\sum_{t=2}^{T} Z^2_{i,t-1}\right)^{-1}. \tag{A46}$$

Because of Assumption (L1*viii*), $\mathrm{E}[Z^2_{i,t-1}]$ and $\mathrm{E}[(Z_{i,t-1}v_{it})^2]$ are finite, and

$$\mathrm{E}[Z_{i,t-1}v_{it}] = \mathrm{E}[\mathrm{E}[Z_{i,t-1}v_{it}|Z_{i,t-1}]] = \mathrm{E}[Z_{i,t-1}\mathrm{E}[v_{it}|Z_{i,t-1}]] = 0, \quad i \in \mathbb{N}.$$

The summability, uniform over i, of MA coefficients implies the summability, uniform over i, of the autocovariances of the Z_{it}'s, hence the ergodicity, for all i, of $\{Z_{it}|t \in \mathbb{Z}\}$. Therefore, the denominator of (A46) is such that

$$\left|\frac{1}{T}\sum_{t=2}^{T} Z^2_{i,t-1} - \mathrm{E}[Z^2_{i,t-1}]\right| = o_{\mathrm{P}}(1). \tag{A47}$$

Turning to the numerator, note that $\mathrm{E}[Z_{i,t-1}v_{it}|v_{i,t-1}] = 0$ for all i, so that $\{Z_{i,t-1}v_{it}\}$ is a martingale difference sequence; moreover, because of finite fourth moments and summability of the MA coefficients, it is uniformly integrable (see Proposition 7.7 in Hamilton, 1994). Therefore, by Theorem 19.8 in Davidson (1994), we have ergodicity and therefore, for all i,

$$\left|\frac{1}{T}\sum_{t=2}^{T}(Z_{i,t-1}v_t)^2 - \mathrm{E}[(Z_{i,t-1}v_t)^2]\right| = o_{\mathrm{P}}(1).$$

This, along with weak stationarity, implies that all conditions for the central limit theorem for martingale differences, as stated, for instance, in Theorem 24.3 of Davidson (1994), hold, yielding, for all i,

$$\left|\frac{1}{T}\sum_{t=2}^{T} Z_{i,t-1}v_{it} - \mathrm{E}[Z_{i,t-1}v_{it}]\right| = O_{\mathrm{P}}\left(1/\sqrt{T}\right). \tag{A48}$$

Going back to (A46), (A47) and (A48) entail $\max_{i=1,\ldots,n}|\widetilde{c}_{i1}-c_{i1}| = O_{\mathrm{P}}\big(1/\sqrt{T}\big)$; therefore, from (A43), (A44), (A45), and (A48),

$$\max_{i=1,\ldots,n}|\widehat{c}_{i1}-c_{i1}| \leq \max_{i=1,\ldots,n}|\widehat{c}_{i1}-\widetilde{c}_{i1}| + \max_{i=1,\ldots,n}|\widetilde{c}_{i1}-c_{i1}| = O_{\mathrm{P}}(\rho_{nT}\log^2 T), \tag{A49}$$

which in turn implies part *(c)* of the proposition. Last, defining $\widehat{v}_{it} := \widehat{Z}_{it} - \widehat{c}_{i1}\widehat{Z}_{i,t-1}$, we have, in view of (A42) and (A49),

$$\max_{i=1,\ldots,n}\max_{t=1,\ldots,T}|\widehat{v}_{it}-v_{it}| = O_{\mathrm{P}}(\rho_{nT}\log^2 T). \tag{A50}$$

Part *(d)* of the proposition follows. □

A.4 Proof of Proposition 2

It follows from Proposition 1 parts *(b)* (see also (A41)) and *(d)* (see also (A50)) that

$$\max_{i=1,\dots,n}\max_{t=1,\dots,T}|(\widehat{e}_{it}+\widehat{v}_{it})-(e_{it}+v_{it})|=\max_{i=1,\dots,n}\max_{t=1,\dots,T}|\widehat{s}_{it}-s_{it}|=O_{\mathrm{P}}(\rho_{nT}\log^2 T). \tag{A51}$$

Assumption (R) implies that, for any i and any $t\in\mathcal{T}^c_{i;nT}:=\{1,\dots,T\}\setminus\mathcal{T}_{i;nT}$,

$$|\widehat{h}_{it}-h_{it}|=|\log\widehat{s}^2_{it}-\log s^2_{it}|=2|\log|\widehat{s}_{it}|-\log|s_{it}||\le\frac{2}{\kappa_T}|\widehat{s}_{it}-s_{it}|. \tag{A52}$$

From (A51) and (A52) we obtain

$$\max_{i=1,\dots,n}\max_{t\in\mathcal{T}^c_{i;nT}}|\widehat{h}_{it}-h_{it}|=O_{\mathrm{P}}\left(\rho_{nT}\log^2 T/\kappa_T\right). \tag{A53}$$

Hereafter, let $\mathbb{T}_{ij;nT}:=\mathcal{T}_{i;nT}\cup\mathcal{T}_{j,nT}$. Denoting by $\widehat{\gamma}^h_{ijk}$ the oracle estimators (computed from the unavailable $\mathbf{h}_n$ values) of $\mathbf{h}_n$'s lag k cross-covariances and by $\widehat{\gamma}^{\widehat{h}}_{ijk}$ the estimator obtained by plugging in the estimated values $\widehat{\mathbf{h}}_n$ of $\mathbf{h}_n$ for the actual ones, we have, for any i,j,k,

$$\begin{aligned}|\widehat{\gamma}^{\widehat{h}}_{ijk}-\widehat{\gamma}^h_{ijk}|\le&\left|\frac{1}{T}\sum_{\substack{t=k+1\\ t,(t-k)\in\mathbb{T}^c_{ij;nT}}}^{T}\left(\widehat{h}_{it}\widehat{h}_{jt-k}-h_{it}h_{jt-k}\right)\right|+\left|\frac{1}{T}\sum_{\substack{t=k+1\\ t,(t-k)\in\mathbb{T}_{ij;nT}}}^{T}\left(\widehat{h}_{it}\widehat{h}_{jt-k}-h_{it}h_{jt-k}\right)\right|\\&+\left|\frac{1}{T}\sum_{\substack{t=k+1\\ t\in\mathbb{T}_{ij;nT}\\(t-k)\in\mathbb{T}^c_{ij;nT}}}^{T}\left(\widehat{h}_{it}\widehat{h}_{jt-k}-h_{it}h_{jt-k}\right)\right|+\left|\frac{1}{T}\sum_{\substack{t=k+1\\ t\in\mathbb{T}^c_{ij;nT}\\(t-k)\in\mathbb{T}_{ij;nT}}}^{T}\left(\widehat{h}_{it}\widehat{h}_{jt-k}-h_{it}h_{jt-k}\right)\right|\\=&DI+DII+DIII+DIV,\quad\text{say.}\end{aligned} \tag{A54}$$

Considering term DI first,

$$\begin{aligned}DI\le&\left|\frac{1}{T}\sum_{\substack{t=k+1\\ t,(t-k)\mathbb{T}^c_{ij;nT}}}^{T}\left(\widehat{h}_{it}-h_{it}\right)h_{jt-k}\right|+\left|\frac{1}{T}\sum_{\substack{t=k+1\\ t,(t-k)\in\mathbb{T}^c_{ij;nT}}}^{T}\left(\widehat{h}_{jt-k}-h_{jt-k}\right)h_{it}\right|\\&+\left|\frac{1}{T}\sum_{\substack{t=k+1\\ t,(t-k)\in\mathbb{T}^c_{ij;nT}}}^{T}\left(\widehat{h}_{it}-h_{it}\right)\left(\widehat{h}_{jt-k}-h_{jt-k}\right)\right|=DI_A+DI_B+DI_C,\quad\text{say.}\end{aligned} \tag{A55}$$

For DI_A we have (recall that $\mathbb{T}^c_{ij;nT}=\mathcal{T}^c_{i;nT}\cap\mathcal{T}^c_{j,nT}$)

$$DI_A\le\max_{t\in\mathbb{T}^c_{ij;nT}}|\widehat{h}_{it}-h_{it}|\left|\frac{1}{T}\sum_{\substack{t=k+1\\(t-k)\in\mathbb{T}^c_{ij;nT}}}^{T}h_{jt-k}\right|\le\max_{t=1,\dots,T}|\widehat{h}_{it}-h_{it}|\max_{t=1,\dots,T}|h_{jt-k}|, \tag{A56}$$

since $|\mathbb{T}^c_{ij;nT}|/T\le 1$. Note that (A56) holds independently of all k such that $|k|\le M_T$. Then, since $\|\boldsymbol{\ell}_j\|=1$, by Assumption (T*iv*) we have that

$$\max_{j=1,\dots,n}\|\xi_{jt}\|_{\psi_1}=\max_{j=1,\dots,n}\|\boldsymbol{\ell}'_j\boldsymbol{\xi}_t\|_{\psi_1}\le\sup_{\boldsymbol{w}_n:\|\boldsymbol{w}_n\|=1}\|\boldsymbol{w}'_n\boldsymbol{\xi}_t\|_{\psi_1}\le K_\xi,$$

for all $n\in\mathbb{N}$. Therefore, using also Assumptions (V1*ii*) and (T*ii*), following the same steps leading to (A35) and (A36), we have

$$\max_{j=1,\dots,n}\max_{t=1,\dots,T}|h_{jt}|\le M_5\max_{j=1,\dots Q}\max_{t=1,\dots,T}|\varepsilon_{jt}|+\max_{j=1,\dots,n}\max_{t=1,\dots,T}|\xi_{jt}|=O_{\mathrm{P}}(\log T). \tag{A57}$$

Substituting (A53) and (A57) into (A56) we have $DI_A=O_{\mathrm{P}}\left(\rho_{nT}\log^3 T/\kappa_T\right)$, uniformly over i,j,k. Terms DI_B and DI_C can be treated similarly, and therefore $DI=O_{\mathrm{P}}\left(\rho_{nT}\log^3 T/\kappa_T\right)$, uniformly over i,j,k. Turning to DII, first notice that, for $t \in \mathcal{T}_{i;nT}$, we have $\widehat{h}_{it} = \log(\kappa_T^2)$ for all i. Then, DII is bounded from above by $DII_A + DII_B + DII_C$, where, as in (A56), because of Assumption (R), we have

$$\begin{aligned}
DII_A &\le \left[\frac{|\mathbb{T}_{ij;nT}|}{T}\Big(\max_{t\in\mathbb{T}_{ij;nT}} |\widehat{h}_{it} - h_{it}|\Big)\right]\Big(\max_{t=1,\dots,T} |h_{jt}|\Big)\\
&\le \left[\frac{2\max_{i=1,\dots,n}|\mathcal{T}_{i;nT}|}{T}\Big(|\log(\kappa_T^2)| + \max_{t\in\mathbb{T}_{ij;nT}} |h_{it}|\Big)\right]\Big(\max_{t=1,\dots,T} |h_{jt}|\Big)\\
&\le \left[o_{\mathrm{P}}\left(\frac{1}{\sqrt{T}}\right)\Big(|\log(\kappa_T^2)| + \max_{t\in\mathbb{T}_{ij;nT}} |h_{it}|\Big)\right]\Big(\max_{t=1,\dots,T} |h_{jt}|\Big)\\
&= o_{\mathrm{P}}\left(\frac{|\log\kappa_T^2|\log T}{\sqrt{T}}\right) + o_{\mathrm{P}}\left(\frac{\log^2 T}{\sqrt{T}}\right) = o_{\mathrm{P}}\left(\frac{|\log\kappa_T^2|\log T}{\sqrt{T}}\right) + o_{\mathrm{P}}(\rho_{nT}\log^3 T), \qquad \text{(A58)}
\end{aligned}$$

uniformly over i, j, k, because of (A57). Terms DII_B and DII_C are analogous to terms DI_B and DI_C, and can be treated similarly. It follows that $II = o_{\mathrm{P}}\left(|\log\kappa_T^2|\log T/\sqrt{T}\right) + o_{\mathrm{P}}(\rho_{nT}\log^3 T)$. The same result can be obtained along similar lines for terms $DIII$ and DIV.

Therefore, from (A54), and since κ_T is of order $\log^{-\varphi} T$ by Assumption (R), we obtain

$$\begin{aligned}
\max_{i,j=1,\dots,n}\max_{|k|\le M_T} \left|\widehat{\gamma}^{\widehat{h}}_{ijk} - \widehat{\gamma}^{h}_{ijk}\right| &= O_{\mathrm{P}}\left(\rho_{nT}\log^3 T/\kappa_T\right) + o_{\mathrm{P}}\left(|\log\kappa_T^2|\log T/\sqrt{T}\right) + o_{\mathrm{P}}(\rho_{nT}\log^3 T)\\
&= O_{\mathrm{P}}(\rho_{nT}\log^{3+\varphi} T) + o_{\mathrm{P}}\left(\log\log T\log T/\sqrt{T}\right) = O_{\mathrm{P}}(\rho_{nT}\log^{3+\varphi} T). \qquad \text{(A59)}
\end{aligned}$$

Now let $\widehat{\sigma}^{\widehat{h}}_{ij}(\theta_\ell)$ be the (i,j)-th entry of the estimated spectral density computed from $\widehat{\mathbf{h}}_n$: then, using (A59) and the definition of the Bartlett kernel, we have

$$\begin{aligned}
\max_{i,j=1,\dots,n}\max_{|\ell|\le M_T} \left|\widehat{\sigma}^{\widehat{h}}_{ij}(\theta_\ell) - \widehat{\sigma}^{h}_{ij}(\theta_\ell)\right| &= \max_{i,j=1,\dots,n}\max_{|\ell|\le M_T}\left|\frac{1}{2\pi}\sum_{k=-T+1}^{T-1} \mathrm{K}\left(\frac{k}{M_T}\right)e^{-ik\theta_\ell}\left(\widehat{\gamma}^{\widehat{h}}_{ijk} - \widehat{\gamma}^{h}_{ijk}\right)\right|\\
&\le \frac{1}{2\pi}\sum_{|k|\le M_T}\left(1 - \frac{k}{M_T}\right)\max_{i,j=1,\dots,n}\left|\widehat{\gamma}^{\widehat{h}}_{ijk} - \widehat{\gamma}^{h}_{ijk}\right|\\
&\le \frac{(2M_T+1)}{2\pi}\max_{i,j=1,\dots,n}\max_{|k|\le M_T}\left|\widehat{\gamma}^{\widehat{h}}_{ijk} - \widehat{\gamma}^{h}_{ijk}\right| = O_{\mathrm{P}}(M_T\rho_{nT}\log^{3+\varphi} T).
\end{aligned}$$

Therefore,

$$\max_{|\ell|\le M_T}\frac{1}{n}\left\|\widehat{\mathbf{\Sigma}}^{\widehat{h}}_n(\theta_\ell) - \widehat{\mathbf{\Sigma}}^{h}_n(\theta_h)\right\| = O_{\mathrm{P}}(M_T\rho_{nT}\log^{3+\varphi} T). \qquad \text{(A60)}$$

Moreover, from Lemma A1(ii), we have the following (see also Lemma 1 in Forni et al., 2017)

$$\mathrm{E}\left[\max_{|\ell|\le M_T}\frac{1}{n^2}\left\|\widehat{\mathbf{\Sigma}}^{h}_n(\theta_\ell) - \mathbf{\Sigma}^{h}_n(\theta_\ell)\right\|^2\right] \le \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n \mathrm{E}\left[\max_{|\ell|\le M_T}\left|\widehat{\sigma}^{h}_{ij}(\theta_\ell) - \sigma^{h}_{ij}(\theta_\ell)\right|^2\right] \le C_3 M_T^2/T + C_4/M_T^2,$$

and, by Chebychev's inequality,

$$\max_{|\ell|\le M_T}\frac{1}{n}\left\|\widehat{\mathbf{\Sigma}}^{h}_n(\theta_\ell) - \mathbf{\Sigma}^{h}_n(\theta_\ell)\right\| = O_{\mathrm{P}}\left(\max\left(M_T/\sqrt{T}, 1/M_T\right)\right). \qquad \text{(A61)}$$

From (A60), (A61) and Lemma 3(i), and, since $M_T\rho_{nT} = \tau_{nT}$,

$$\begin{aligned}
\max_{|\ell|\le M_T}\frac{1}{n}\left\|\widehat{\mathbf{\Sigma}}^{\widehat{h}}_n(\theta_\ell) - \mathbf{\Sigma}^{\chi}_n(\theta_\ell)\right\| &\le \max_{|\ell|\le M_T}\frac{1}{n}\left\|\widehat{\mathbf{\Sigma}}^{\widehat{h}}_n(\theta_\ell) - \widehat{\mathbf{\Sigma}}^{h}_n(\theta_\ell)\right\| + \max_{|\ell|\le M_T}\frac{1}{n}\left\|\widehat{\mathbf{\Sigma}}^{h}_n(\theta_\ell) - \mathbf{\Sigma}^{h}_n(\theta_\ell)\right\| + \max_{|\ell|\le M_T}\frac{1}{n}\left\|\mathbf{\Sigma}^{\xi}_n(\theta_\ell)\right\|\\
&= O_{\mathrm{P}}\left(\max\left(\tau_{nT}\log^{3+\varphi} T, M_T/\sqrt{T}, 1/M_T, 1/n\right)\right) = O_{\mathrm{P}}(\tau_{nT}\log^{3+\varphi} T).
\end{aligned}$$

and, following the same steps leading to (A22), we can prove that:

$$\max_{i,j=1,\dots,n}\max_{|\ell|\le M_T}\left|\widehat{\sigma}^{\chi}_{ij}(\theta_\ell) - \sigma^{\chi}_{ij}(\theta_\ell)\right| = O_{\mathrm{P}}(\tau_{nT}\log^{3+\varphi} T). \qquad \text{(A62)}$$

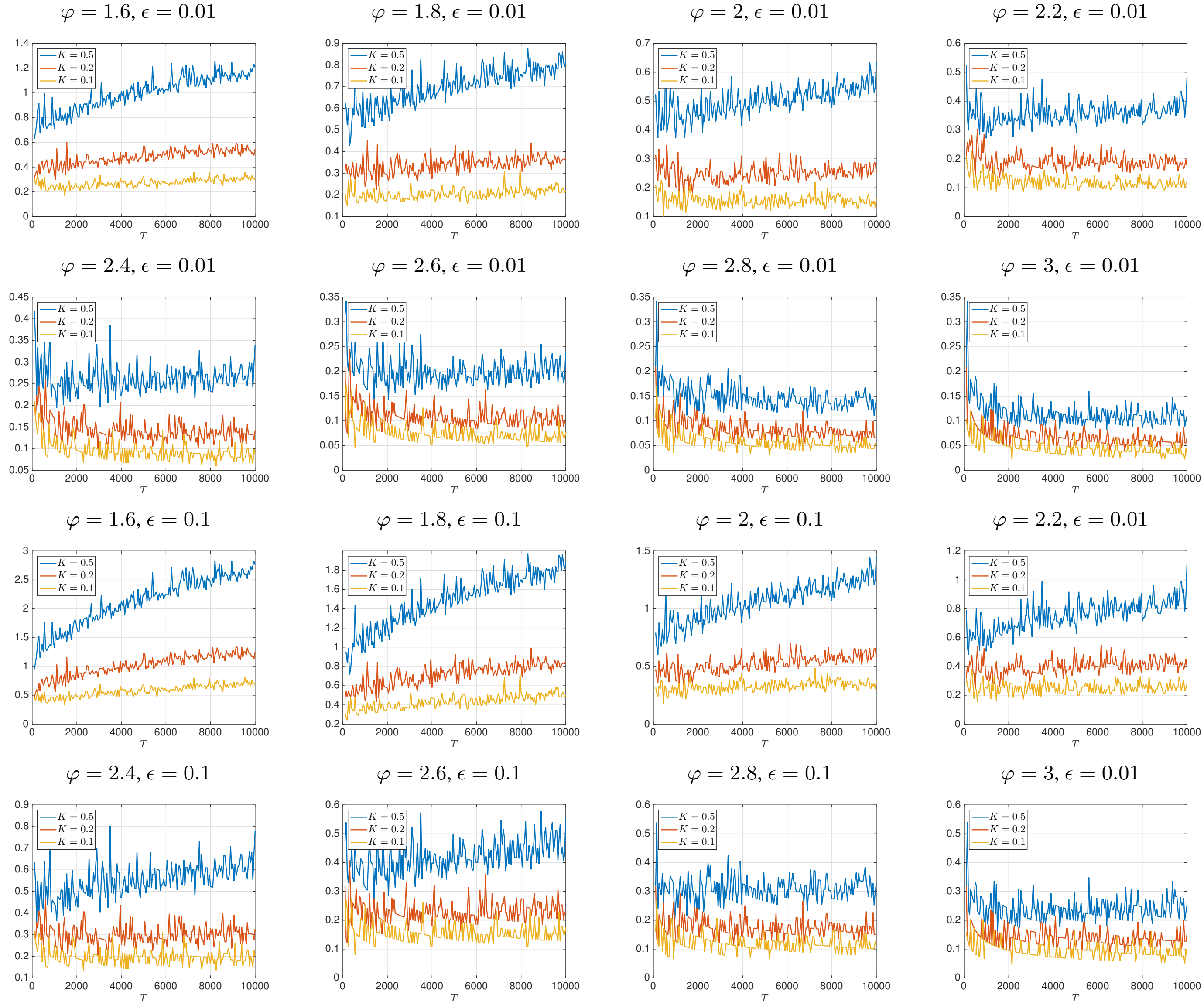
From there on, the proof of Proposition 2 is strictly identical to that of Proposition 1. In particular: for part *(a)* we have the same rate of convergence as in (A62); for part *(b)* we have an additional $\log T$ term, by following the same reasoning leading to the bound of (A29)); and for parts *(c)* and *(d)* we have one more $\log T$ term, by following the same reasoning leading to (A49) and (A50), respectively. □

B Assumption (R): empirical evidence

In this section, we provide some empirical evidence that Assumption (R) holds in the S&P100 panel under study. From the estimated nT realisations of the estimated panel $\{\widehat{\mathbf{s}}_t\}$ obtained in the previous section, we let the sample size vary and we simulate M artificial datasets $\{\widehat{\mathbf{s}}_t^*\}$ of size $N \times T_j$, $j = 1, \ldots, M$, by uniformly sampling with replacement from $\{\widehat{\mathbf{s}}_t\}$. In particular, we consider $M = 199$ different sample sizes such that $T_1 = 100$, $T_M = 10000$ and $T_j = T_{j-1} + 50$ for $j = 2, \ldots, M$; for each value of T_j, we simulate $N = 150$ time series.

We set $\kappa_T = K/\log^{\varphi} T$, and for each given T_j we compute the cardinality of the set $\mathcal{T}_{i,NT_j}$. If Assumption (R) holds, then the quantity $r(j, \varphi, K, \epsilon) := \max_{i=1,\ldots,N} T^{\epsilon} |\mathcal{T}_{i,NT_j}| / \sqrt{T_j}$ should tend to a constant as T_j grows, for any $\epsilon > 0$. In Figure B1, we report $r(j, \varphi, K, \epsilon)$, as function of T_j, when $\epsilon \in \{0.01, 0.1\}$, $\varphi \in \{1.6, 1.8, 2, 2.2, 2.4, 2.6, 2.8, 3\}$ and $K \in \{0.5, 0.2, 0.1\}$.

FIGURE B1: *Large sample behaviour of $r(j, \varphi, K, \epsilon)$ as a function of T_j*



C S&P100 data

TABLE C1: *S&P100 constituents.*

Ticker	Name		
AAPL	Apple Inc.	HPQ	Hewlett Packard Co.
ABT	Abbott Laboratories	IBM	International Business Machines
AEP	American Electric Power Co.	INTC	Intel Corporation
AIG	American International Group Inc.	JNJ	Johnson & Johnson Inc.
ALL	Allstate Corp.	JPM	JP Morgan Chase & Co.
AMGN	Amgen Inc.	KO	The Coca-Cola Company
AMZN	Amazon.com	LLY	Eli Lilly and Company
APA	Apache Corp.	LMT	Lockheed-Martin
APC	Anadarko Petroleum Corp.	LOW	Lowe's
AXP	American Express Inc.	MCD	McDonald's Corp.
BA	Boeing Co.	MDT	Medtronic Inc.
BAC	Bank of America Corp.	MMM	3M Company
BAX	Baxter International Inc.	MO	Altria Group
BK	Bank of New York	MRK	Merck & Co.
BMY	Bristol-Myers Squibb	MS	Morgan Stanley
BRK.B	Berkshire Hathaway	MSFT	Microsoft
C	Citigroup Inc.	NKE	Nike
CAT	Caterpillar Inc.	NOV	National Oilwell Varco
CL	Colgate-Palmolive Co.	NSC	Norfolk Southern Corp.
CMCSA	Comcast Corp.	ORCL	Oracle Corporation
COF	Capital One Financial Corp.	OXY	Occidental Petroleum Corp.
COP	ConocoPhillips	PEP	Pepsico Inc.
COST	Costco	PFE	Pfizer Inc.
CSCO	Cisco Systems	PG	Procter & Gamble Co.
CVS	CVS Caremark	QCOM	Qualcomm Inc.
CVX	Chevron	RTN	Raytheon Co.
DD	DuPont	SBUX	Starbucks Corporation
DELL	Dell	SLB	Schlumberger
DIS	The Walt Disney Company	SO	Southern Company
DOW	Dow Chemical	SPG	Simon Property Group, Inc.
DVN	Devon Energy	T	AT&T Inc.
EBAY	eBay Inc.	TGT	Target Corp.
EMC	EMC Corporation	TWX	Time Warner Inc.
EMR	Emerson Electric Co.	TXN	Texas Instruments
EXC	Exelon	UNH	UnitedHealth Group Inc.
F	Ford Motor	UNP	Union Pacific Corp.
FCX	Freeport-McMoran	UPS	United Parcel Service Inc.
FDX	FedEx	USB	US Bancorp
GD	General Dynamics	UTX	United Technologies Corp.
GE	General Electric Co.	VZ	Verizon Communications Inc.
GILD	Gilead Sciences	WAG	Walgreens
GS	Goldman Sachs	WFC	Wells Fargo
HAL	Halliburton	WMB	Williams Companies
HD	Home Depot	WMT	Wal-Mart
HON	Honeywell	XOM	Exxon Mobil Corp.

D Additional simulation results

TABLE D1: *Simulation results. Common components. Values of the bandwidths are:* $B_T = 5$ *and* $M_T = 5$ *for all* T.

$q = 1, Q = 1$

		$T = 200$		$T = 500$		$T = 1000$	
		$n = 100$	$n = 200$	$n = 100$	$n = 200$	$n = 100$	$n = 200$
MSE^X		0.089	0.091	0.073	0.075	0.061	0.064
MSE^χ	$\kappa_T = 0$	0.450	0.412	0.630	0.607	0.793	0.734
MSE^χ	$\kappa_T = 0.2$	0.469	0.448	0.673	0.641	0.750	0.730
MSE^χ	$\kappa_T = 0.4$	0.455	0.477	0.671	0.672	0.788	0.820
MAD^X		0.184	0.167	0.176	0.158	0.170	0.157
MAD^χ	$\kappa_T = 0$	0.486	0.458	0.575	0.555	0.650	0.619
MAD^χ	$\kappa_T = 0.2$	0.491	0.468	0.591	0.563	0.620	0.608
MAD^χ	$\kappa_T = 0.4$	0.481	0.487	0.583	0.579	0.630	0.637
MAX^X		9.016	10.336	11.809	10.955	11.742	16.046
MAX^χ	$\kappa_T = 0$	11.385	19.027	18.554	40.420	55.108	50.038
MAX^χ	$\kappa_T = 0.2$	18.154	29.037	34.038	37.224	45.037	61.851
MAX^χ	$\kappa_T = 0.4$	20.399	23.017	33.477	44.226	65.348	51.665

$q = 3, Q = 2$

		$T = 200$		$T = 500$		$T = 1000$	
		$n = 100$	$n = 200$	$n = 100$	$n = 200$	$n = 100$	$n = 200$
MSE^X		0.091	0.096	0.067	0.070	0.056	0.058
MSE^χ	$\kappa_T = 0$	0.361	0.277	0.373	0.259	0.420	0.257
MSE^χ	$\kappa_T = 0.2$	0.316	0.232	0.340	0.336	0.229	0.225
MSE^χ	$\kappa_T = 0.4$	0.300	0.226	0.305	0.210	0.319	0.218
MAD^X		0.213	0.205	0.186	0.183	0.175	0.172
MAD^χ	$\kappa_T = 0$	0.458	0.395	0.461	0.380	0.489	0.376
MAD^χ	$\kappa_T = 0.2$	0.426	0.359	0.439	0.432	0.354	0.350
MAD^χ	$\kappa_T = 0.4$	0.413	0.354	0.413	0.338	0.421	0.344
MAX^X		5.632	8.485	12.693	8.876	11.027	13.225
MAX^χ	$\kappa_T = 0$	5.151	4.585	5.254	6.031	7.377	5.736
MAX^χ	$\kappa_T = 0.2$	3.897	4.985	4.981	6.400	5.038	6.491
MAX^χ	$\kappa_T = 0.4$	5.855	4.454	7.723	7.237	5.837	5.723

$q = 2, Q = 3$

		$T = 200$		$T = 500$		$T = 1000$	
		$n = 100$	$n = 200$	$n = 100$	$n = 200$	$n = 100$	$n = 200$
MSE^X		0.087	0.088	0.068	0.071	0.056	0.064
MSE^χ	$\kappa_T = 0$	0.355	0.300	0.384	0.274	0.425	0.280
MSE^χ	$\kappa_T = 0.2$	0.327	0.265	0.347	0.236	0.380	0.231
MSE^χ	$\kappa_T = 0.4$	0.300	0.247	0.315	0.208	0.361	0.215
MAD^X		0.210	0.204	0.191	0.187	0.177	0.180
MAD^χ	$\kappa_T = 0$	0.459	0.419	0.475	0.396	0.496	0.399
MAD^χ	$\kappa_T = 0.2$	0.439	0.391	0.450	0.366	0.469	0.361
MAD^χ	$\kappa_T = 0.4$	0.420	0.376	0.427	0.343	0.456	0.347
MAX^X		9.532	9.098	6.292	11.581	8.078	7.409
MAX^χ	$\kappa_T = 0$	3.978	4.204	4.184	5.321	6.107	5.417
MAX^χ	$\kappa_T = 0.2$	3.986	4.835	4.913	5.932	5.512	7.808
MAX^χ	$\kappa_T = 0.4$	4.524	4.674	4.521	6.284	5.944	5.705

TABLE D2: *Simulation results. Empirical coverage and frequency of confidence bound violations averaged over all n series and all $\mathcal{M}$ replications, when $T = 1000$ and $\mathcal{M} = 200$. Values of the bandwidths are: $B_T = 5$ and $M_T = 5$ for all T.*

$q = 1, Q = 1$

		$n = 100$					$n = 200$				
		α					α				
		0.32	0.2	0.1	0.05	0.01	0.32	0.2	0.1	0.05	0.01
$C(\alpha)$	$\kappa_T = 0$	0.6854	0.8098	0.9099	0.9581	0.9928	0.6793	0.7871	0.8809	0.9335	0.9831
$V_+(\alpha/2)$		0.1573	0.0951	0.0441	0.0217	0.0040	0.1606	0.1055	0.0599	0.0332	0.0087
$V_-(\alpha/2)$		0.1573	0.0951	0.0460	0.0202	0.0032	0.1601	0.1074	0.0593	0.0333	0.0083
$C(\alpha)$	$\kappa_T = 0.2$	0.7046	0.7982	0.8790	0.9246	0.9744	0.7126	0.8207	0.9070	0.9490	0.9841
$V_+(\alpha/2)$		0.1465	0.0992	0.0594	0.0373	0.0129	0.1439	0.0891	0.0461	0.0247	0.0081
$V_-(\alpha/2)$		0.1489	0.1026	0.0616	0.0381	0.0127	0.1436	0.0902	0.0469	0.0264	0.0078
$C(\alpha)$	$\kappa_T = 0.4$	0.7531	0.8440	0.9197	0.9543	0.9845	0.7687	0.8511	0.9243	0.9628	0.9969
$V_+(\alpha/2)$		0.1217	0.0759	0.0401	0.0229	0.0084	0.1171	0.0775	0.0406	0.0201	0.0017
$V_-(\alpha/2)$		0.1252	0.0801	0.0402	0.0228	0.0071	0.1142	0.0714	0.0351	0.0172	0.0014

$q = 3, Q = 2$

		$n = 100$					$n = 200$				
		α					α				
		0.32	0.2	0.1	0.05	0.01	0.32	0.2	0.1	0.05	0.01
$C(\alpha)$	$\kappa_T = 0$	0.6821	0.7978	0.8907	0.9405	0.9800	0.6651	0.7872	0.8903	0.9445	0.9840
$V_+(\alpha/2)$		0.1596	0.1003	0.0556	0.0294	0.0097	0.1596	0.1003	0.0556	0.0294	0.0097
$V_-(\alpha/2)$		0.1583	0.1019	0.0537	0.0301	0.0103	0.1583	0.1019	0.0537	0.0301	0.0103
$C(\alpha)$	$\kappa_T = 0.2$	0.6837	0.7787	0.8609	0.9089	0.9602	0.7277	0.8290	0.9124	0.9552	0.9910
$V_+(\alpha/2)$		0.1560	0.1097	0.0689	0.0442	0.0186	0.1387	0.0865	0.0442	0.0220	0.0037
$V_-(\alpha/2)$		0.1603	0.1116	0.0702	0.0469	0.0212	0.1336	0.0846	0.0435	0.0229	0.0054
$C(\alpha)$	$\kappa_T = 0.4$	0.7500	0.8369	0.9141	0.9527	0.9881	0.7551	0.8431	0.9149	0.9543	0.9915
$V_+(\alpha/2)$		0.1217	0.0794	0.0419	0.0233	0.0059	0.1232	0.0782	0.0409	0.0219	0.0036
$V_-(\alpha/2)$		0.1283	0.0837	0.0440	0.0240	0.0060	0.1218	0.0788	0.0442	0.0238	0.0049

$q = 2, Q = 3$

		$n = 100$					$n = 200$				
		α					α				
		0.32	0.2	0.1	0.05	0.01	0.32	0.2	0.1	0.05	0.01
$C(\alpha)$	$\kappa_T = 0$	0.6137	0.7191	0.8259	0.8916	0.9598	0.6631	0.7803	0.8820	0.9393	0.9866
$V_+(\alpha/2)$		0.2034	0.1490	0.0894	0.0552	0.0217	0.1722	0.1118	0.0610	0.0317	0.0078
$V_-(\alpha/2)$		0.1829	0.1319	0.0847	0.0532	0.0185	0.1648	0.1080	0.0571	0.0290	0.0057
$C(\alpha)$	$\kappa_T = 0.2$	0.7166	0.8219	0.9095	0.9549	0.9925	0.7259	0.8332	0.9262	0.9660	0.9941
$V_+(\alpha/2)$		0.1495	0.0940	0.0493	0.0252	0.0040	0.1371	0.0826	0.0372	0.0176	0.0029
$V_-(\alpha/2)$		0.1339	0.0841	0.0412	0.0199	0.0035	0.1371	0.0843	0.0367	0.0164	0.0031
$C(\alpha)$	$\kappa_T = 0.4$	0.7715	0.8569	0.9282	0.9671	0.9950	0.7265	0.8165	0.8986	0.9466	0.9918
$V_+(\alpha/2)$		0.1146	0.0728	0.0361	0.0149	0.0019	0.1232	0.0782	0.0409	0.0219	0.0036
$V_-(\alpha/2)$		0.1139	0.0703	0.0357	0.0180	0.0031	0.1218	0.07878	0.0442	0.0238	0.0049

TABLE D3: *Simulation results. Common components. Values of the bandwidths are:* $B_T = 1$ *and* $M_T = 14$ *for* $T = 200$*;* $B_T = 1$ *and* $M_T = 22$ *for* $T = 500$*;* $B_T = 1$ *and* $M_T = 31$ *for* $T = 1000$*.*

$q = 1, Q = 1$

		$T = 200$		$T = 500$		$T = 1000$	
		$n = 100$	$n = 200$	$n = 100$	$n = 200$	$n = 100$	$n = 200$
MSE^X		0.467	4.568	0.334	0.344	0.271	0.251
MSE^χ	$\kappa_T = 0$	0.547	0.508	0.477	0.422	0.419	0.343
MSE^χ	$\kappa_T = 0.2$	0.561	0.527	0.481	0.427	0.412	0.331
MSE^χ	$\kappa_T = 0.4$	0.593	0.487	0.405	0.560	0.458	0.350
MAD^X		0.417	0.386	0.366	0.330	0.337	0.294
MAD^χ	$\kappa_T = 0$	0.555	0.524	0.508	0.467	0.471	0.417
MAD^χ	$\kappa_T = 0.2$	0.556	0.526	0.503	0.460	0.459	0.399
MAD^χ	$\kappa_T = 0.4$	0.573	0.502	0.448	0.541	0.472	0.406
MAX^X		36.332	826.833	20.032	39.123	31.799	46.432
MAX^χ	$\kappa_T = 0$	7.023	8.756	7.328	8.536	8.251	8.111
MAX^χ	$\kappa_T = 0.2$	8.059	8.743	7.758	9.147	9.171	8.843
MAX^χ	$\kappa_T = 0.4$	8.727	9.109	9.846	9.310	9.387	9.895

$q = 3, Q = 2$

		$T = 200$		$T = 500$		$T = 1000$	
		$n = 100$	$n = 200$	$n = 100$	$n = 200$	$n = 100$	$n = 200$
MSE^X		0.255	0.250	0.159	0.157	0.116	0.121
MSE^χ	$\kappa_T = 0$	0.389	0.375	0.299	0.273	0.248	0.234
MSE^χ	$\kappa_T = 0.2$	0.369	0.268	0.212	0.360	0.241	0.202
MSE^χ	$\kappa_T = 0.4$	0.358	0.358	0.250	0.252	0.191	0.176
MAD^X		0.357	0.337	0.284	0.272	0.248	0.245
MAD^χ	$\kappa_T = 0$	0.480	0.467	0.417	0.395	0.380	0.365
MAD^χ	$\kappa_T = 0.2$	0.465	0.391	0.348	0.454	0.367	0.334
MAD^χ	$\kappa_T = 0.4$	0.457	0.450	0.376	0.372	0.328	0.311
MAX^X		8.732	10.161	10.535	11.145	12.495	13.895
MAX^χ	$\kappa_T = 0$	5.721	5.525	5.010	5.412	4.532	5.946
MAX^χ	$\kappa_T = 0.2$	6.036	5.137	4.693	5.637	5.194	6.250
MAX^χ	$\kappa_T = 0.4$	5.332	5.653	5.659	6.776	5.285	7.061

$q = 2, Q = 3$

		$T = 200$		$T = 500$		$T = 1000$	
		$n = 100$	$n = 200$	$n = 100$	$n = 200$	$n = 100$	$n = 200$
MSE^X		0.298	0.286	0.194	0.184	0.134	0.143
MSE^χ	$\kappa_T = 0$	0.480	0.439	0.387	0.353	0.311	0.300
MSE^χ	$\kappa_T = 0.2$	0.463	0.426	0.358	0.327	0.277	0.266
MSE^χ	$\kappa_T = 0.4$	0.464	0.453	0.352	0.341	0.271	0.247
MAD^X		0.393	0.370	0.316	0.303	0.267	0.272
MAD^χ	$\kappa_T = 0$	0.538	0.511	0.478	0.453	0.428	0.417
MAD^χ	$\kappa_T = 0.2$	0.526	0.500	0.455	0.432	0.400	0.388
MAD^χ	$\kappa_T = 0.4$	0.525	0.515	0.449	0.438	0.392	0.372
MAX^X		9.353	10.260	8.710	14.061	9.802	12.255
MAX^χ	$\kappa_T = 0$	6.081	5.367	5.962	6.078	5.683	7.025
MAX^χ	$\kappa_T = 0.2$	5.417	5.818	6.051	6.590	6.481	7.808
MAX^χ	$\kappa_T = 0.4$	5.905	6.056	5.951	7.419	6.519	6.513

TABLE D4: *Simulation results. Empirical coverage and frequency of confidence bound violations averaged over all n series and all $\mathcal{M}$ replications, for $T = 1000$ and $\mathcal{M} = 200$. Values of the bandwidths are: $B_T = 1$ and $M_T = 14$ for $T = 200$; $B_T = 1$ and $M_T = 22$ for $T = 500$; $B_T = 1$ and $M_T = 31$ for $T = 1000$.*

$q = 1, Q = 1$

		$n = 100$					$n = 200$				
				α					α		
		0.32	0.2	0.1	0.05	0.01	0.32	0.2	0.1	0.05	0.01
$C(\alpha)$	$\kappa_T = 0$	0.7211	0.8374	0.9267	0.9708	0.9966	0.6287	0.7476	0.8602	0.9238	0.9856
$V_+(\alpha/2)$		0.1460	0.0855	0.0393	0.0144	0.0020	0.1883	0.1285	0.0703	0.0394	0.0072
$V_-(\alpha/2)$		0.1329	0.0771	0.0340	0.0148	0.0014	0.1831	0.1240	0.0696	0.0368	0.0073
$C(\alpha)$	$\kappa_T = 0.2$	0.7615	0.8597	0.9367	0.9763	0.9975	0.6841	0.7864	0.8846	0.9346	0.9882
$V_+(\alpha/2)$		0.1255	0.0730	0.0332	0.0125	0.0011	0.1603	0.1090	0.0585	0.0335	0.0057
$V_-(\alpha/2)$		0.1130	0.0673	0.0301	0.0112	0.0014	0.1557	0.1047	0.0570	0.0320	0.0062
$C(\alpha)$	$\kappa_T = 0.4$	0.7747	0.8623	0.9368	0.9708	0.9966	0.6957	0.7856	0.8735	0.9272	0.9821
$V_+(\alpha/2)$		0.1135	0.0678	0.0291	0.0133	0.0012	0.1495	0.1060	0.0623	0.0358	0.0081
$V_-(\alpha/2)$		0.1118	0.0699	0.0341	0.0160	0.0014	0.1549	0.1085	0.0642	0.0371	0.0099

$q = 3, Q = 2$

		$n = 100$					$n = 200$				
				α					α		
		0.32	0.2	0.1	0.05	0.01	0.32	0.2	0.1	0.05	0.01
$C(\alpha)$	$\kappa_T = 0$	0.6826	0.8013	0.9040	0.9551	0.9942	0.6327	0.7474	0.8486	0.9090	0.9706
$V_+(\alpha/2)$		0.1552	0.1001	0.0472	0.0229	0.0027	0.1847	0.1263	0.0752	0.0450	0.0142
$V_-(\alpha/2)$		0.1622	0.0986	0.0488	0.0220	0.0031	0.1826	0.1264	0.0763	0.0461	0.0153
$C(\alpha)$	$\kappa_T = 0.2$	0.7226	0.8271	0.9171	0.9594	0.9941	0.6749	0.7732	0.8639	0.9157	0.9720
$V_+(\alpha/2)$		0.1362	0.0869	0.0410	0.0209	0.0031	0.1636	0.1134	0.0672	0.0415	0.0135
$V_-(\alpha/2)$		0.1412	0.0860	0.0419	0.0197	0.0028	0.1616	0.1135	0.0690	0.0428	0.0146
$C(\alpha)$	$\kappa_T = 0.4$	0.7322	0.8250	0.9036	0.9437	0.9850	0.7239	0.8100	0.8913	0.9335	0.9766
$V_+(\alpha/2)$		0.1321	0.0854	0.0470	0.0259	0.0068	0.1392	0.0960	0.0539	0.0333	0.0121
$V_-(\alpha/2)$		0.1357	0.0896	0.0494	0.0304	0.0082	0.1370	0.0941	0.0549	0.0333	0.0114

$q = 2, Q = 3$

		$n = 100$					$n = 200$				
				α					α		
		0.32	0.2	0.1	0.05	0.01	0.32	0.2	0.1	0.05	0.01
$C(\alpha)$	$\kappa_T = 0$	0.6604	0.7730	0.8773	0.9312	0.9767	0.6973	0.8141	0.9118	0.9582	0.9946
$V_+(\alpha/2)$		0.1682	0.1107	0.0595	0.0314	0.0108	0.1481	0.0901	0.0424	0.0204	0.0029
$V_-(\alpha/2)$		0.1714	0.1163	0.0632	0.0374	0.0125	0.1547	0.0959	0.0458	0.0215	0.0026
$C(\alpha)$	$\kappa_T = 0.2$	0.6979	0.8006	0.8899	0.9367	0.9794	0.7446	0.8415	0.9240	0.9647	0.9959
$V_+(\alpha/2)$		0.1507	0.0973	0.0521	0.0288	0.0100	0.1256	0.0759	0.0365	0.0168	0.0022
$V_-(\alpha/2)$		0.1514	0.1021	0.0580	0.0345	0.0106	0.1298	0.0827	0.0396	0.0186	0.0020
$C(\alpha)$	$\kappa_T = 0.4$	0.7368	0.8290	0.9009	0.9462	0.9887	0.7557	0.8420	0.9160	0.9566	0.9902
$V_+(\alpha/2)$		0.1316	0.0864	0.0501	0.0259	0.0056	0.1273	0.0834	0.0434	0.0219	0.0047
$V_-(\alpha/2)$		0.1316	0.0846	0.0490	0.0279	0.0057	0.1171	0.0747	0.0407	0.0216	0.0051

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