Large-Dimensional Dynamic Factor Models: Estimation of Impulse-Response Functions with $I(1)$ Cointegrated Factors

COMPLEMENTARY APPENDIX

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Contents

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Preliminary definitions and notation

Norms. For any $m \times p$ matrix $\mathbf{B}$ with generic element b_{ij}, we denote its spectral norm as $\|\mathbf{B}\| = (\mu_1^{\mathbf{B}'\mathbf{B}})^{1/2}$, where $\mu_1^{\mathbf{B}'\mathbf{B}}$ is the largest eigenvalue of $\mathbf{B}'\mathbf{B}$, the Frobenius norm as $\|\mathbf{B}\|_F = (\text{tr}(\mathbf{B}'\mathbf{B}))^{1/2} = (\sum_i \sum_j b_{ij}^2)^{1/2}$, and the column and row norm as $\|\mathbf{B}\|_1 = \max_j \sum_i |b_{ij}|$ and $\|\mathbf{B}\|_\infty = \max_i \sum_j |b_{ij}|$, respectively. Throughout we make use of the following properties.

1. Subadditivity of the norm, for an $m \times p$ matrix $\mathbf{A}$ and a $p \times s$ matrix $\mathbf{B}$:

$$\|\mathbf{A}\mathbf{B}\| \leq \|\mathbf{A}\| \, \|\mathbf{B}\|. \tag{C1}$$

2. Norm inequalities, for an $n \times n$ symmetric matrix $\mathbf{A}$:

$$\mu_1^A = \|\mathbf{A}\| \leq \sqrt{\|\mathbf{A}\|_1 \, \|\mathbf{A}\|_\infty} = \|\mathbf{A}\|_1, \quad \|\mathbf{A}\| \leq \|\mathbf{A}\|_F, \quad \|\mathbf{A}\|_F \leq \sqrt{n}\|\mathbf{A}\|. \tag{C2}$$

3. Weyl's inequality, for two $n \times n$ symmetric matrices $\mathbf{A}$ and $\mathbf{B}$, with eigenvalues μ_j^A and μ_j^B:

$$|\mu_j^A - \mu_j^B| \leq \|\mathbf{A} - \mathbf{B}\|, \qquad j = 1, \dots, n. \tag{C3}$$

Factors' dynamics. It is convenient to write the dynamic model of the factors, (8), as

$$\Delta F_{jt} = \mathbf{c}_j'(L)\mathbf{u}_t = \sum_{l=1}^{q} c_{jl}(L) u_{lt}, \quad j = 1, \dots r, \tag{C4}$$

where $\mathbf{c}_j(L)$ is an $q \times 1$ infinite rational polynomial matrix with entries $c_{jl}(L)$. Due to rationality, there exists a positive real K_1 such that

$$\sup_{j=1,\dots,r} \sup_{l=1,\dots,q} \sum_{k=0}^{\infty} c_{jlk}^2 \leq K_1. \tag{C5}$$

From Assumption 4 we also have $F_{jt} = \sum_{s=1}^t \mathbf{c}_j'(L)\mathbf{u}_s$.

Idiosyncratic dynamics. Likewise, for the idiosyncratic components it is convenient to write (12) as

$$\Delta \xi_{it} = \check{d}_i(L)\varepsilon_{it}, \quad i = 1, \dots, n, \tag{C6}$$

where $\check{d}_i(L)$ are a infinite polynomials defined as $\check{d}_i(L) = (1-L)(1-\rho_i L)^{-1} d_i(L)$ with $d_i(L)$ also infinite polynomials. Because of Assumption 3(c) there exists a positive real K_2 such that

$$\sup_{i=1,\dots,n} \sum_{k=0}^{\infty} \check{d}_{ik}^2 \leq K_2. \tag{C7}$$

With reference to Assumption 6(a) we have $\rho_i = 1$ if $i \in \mathcal{I}_1$ and $|\rho_i| < 1$ if $i \in \mathcal{I}_1^c$. Hence, by Assumptions 4, we have also $\xi_{it} = \sum_{s=1}^t \check{d}_i(L)\varepsilon_{is}$, which is non-stationary if and only if $i \in \mathcal{I}_1$.

Factors' identification. The following choice of the factors is very convenient and will be adopted in the sequel (see also Remark 3). Let $\mathbf{W}$ be the $n \times r$ matrix whose columns are the right normalised eigenvectors of the variance-covariance matrix of $\Delta\boldsymbol{\chi}_t$, corresponding to the first r eigenvalues $\mu_j^{\Delta\chi}$, $j = 1, \dots, r$. Following Forni et al. (2009) we identify the differenced factors by defining $\Delta\mathbf{F}_t = \mathbf{W}'\Delta\boldsymbol{\chi}_t$. Now project $\Delta\boldsymbol{\chi}_t$ on $\Delta\mathbf{F}_t$: $\Delta\boldsymbol{\chi}_t = \boldsymbol{\mathcal{A}}\Delta\mathbf{F}_t + \boldsymbol{\mathcal{R}}_t$. We see that $\boldsymbol{\mathcal{A}} = \mathbf{W}$ and that the variance-covariance matrices of $\Delta\boldsymbol{\chi}_t$ and of $\mathbf{W}\Delta\mathbf{F}_t$ are equal, so that $\boldsymbol{\mathcal{R}}_t = \mathbf{0}$ and the projection becomes $\Delta\boldsymbol{\chi}_t = \mathbf{W}\mathbf{W}'\Delta\boldsymbol{\chi}_t$, that is $(\mathbf{I}_n - \mathbf{W}\mathbf{W}')\,\Delta\boldsymbol{\chi}_t = \mathbf{0}$. Since, by Assumption 4, $\boldsymbol{\chi}_0 = \mathbf{0}$, we obtain $\boldsymbol{\chi}_t = \mathbf{W}\mathbf{W}'\boldsymbol{\chi}_t$, for $t > 0$, or, in our preferred specification, $\boldsymbol{\chi}_t = [\sqrt{n}\mathbf{W}][n^{-1/2}\mathbf{W}'\boldsymbol{\chi}_t]$. We set henceforth, for all $n \in \mathbb{N}$,

$$\boldsymbol{\Lambda} = \sqrt{n}\mathbf{W}, \quad \mathbf{F}_t = \frac{1}{\sqrt{n}}\mathbf{W}'\boldsymbol{\chi}_t = \frac{1}{n}\boldsymbol{\Lambda}'\boldsymbol{\chi}_t. \tag{C8}$$

Note that now the factors $\mathbf{F}_t$ and the loadings $\boldsymbol{\lambda}_i$, for a given i, depend on n.

Sample size of differenced data. The data in level is assumed to be observed for $t = 1, \dots, T$, thus the sample size is T, which implies that the sample size of the data in differences is $(T-1)$. When both levels and differences are present in the same proof we keep the distinction between the two sample sizes, however, in proofs where no confusion can arise we use just T as sample size.

C Proof of Lemma 1

In order to prove part (i), we first prove results on the asymptotic properties of the sample covariance and of its eigenvalues and eigenvectors.

Sample covariance matrix. From Assumption 3(e) of independent common and idiosyncratic components, we have $\boldsymbol{\Gamma}_0^{\Delta x} = \boldsymbol{\Gamma}_0^{\Delta \chi} + \boldsymbol{\Gamma}_0^{\Delta \xi}$ and therefore from Lemmas D3 (which holds uniformly over all i and j) and D2(ii) and Assumption 3(e) we have

$$\left\| \frac{\widehat{\boldsymbol{\Gamma}}_0^{\Delta y}}{n} - \frac{\boldsymbol{\Gamma}_0^{\Delta \chi}}{n} \right\| \le \left\| \frac{\widehat{\boldsymbol{\Gamma}}_0^{\Delta y}}{n} - \frac{\boldsymbol{\Gamma}_0^{\Delta x}}{n} \right\| + \left\| \frac{\boldsymbol{\Gamma}_0^{\Delta \xi}}{n} \right\| \le \sqrt{\frac{1}{n^2}\sum_{i=1}^n \sum_{j=1}^n \left(\widehat{\gamma}_{ij}^{\Delta y} - \gamma_{ij}^{\Delta x}\right)^2} + \frac{\mu_1^{\Delta \xi}}{n}$$
$$\le O_p\left(\frac{1}{\sqrt{T}}\right) + \frac{M_7}{n} = O_p\left(\max\left(\frac{1}{\sqrt{T}}, \frac{1}{n}\right)\right). \tag{C9}$$

Moreover, by denoting as $\boldsymbol{\epsilon}_i$ an n-dimensional vector with 1 as i-th entry and all other entries equal to zero, again by Lemmas D3 and D2(ii), we have

$$\left\| \frac{\boldsymbol{\epsilon}_i'}{\sqrt{n}}\left(\widehat{\boldsymbol{\Gamma}}_0^{\Delta y} - \boldsymbol{\Gamma}_0^{\Delta \chi}\right) \right\| \le \left\| \frac{\boldsymbol{\epsilon}_i'}{\sqrt{n}}\left(\widehat{\boldsymbol{\Gamma}}_0^{\Delta y} - \boldsymbol{\Gamma}_0^{\Delta x}\right) \right\| + \left\| \frac{\boldsymbol{\epsilon}_i'\boldsymbol{\Gamma}_0^{\Delta \xi}}{\sqrt{n}} \right\| \le \sqrt{\frac{1}{n}\sum_{j=1}^n \left(\widehat{\gamma}_{ij}^{\Delta y} - \gamma_{ij}^{\Delta x}\right)^2} + \frac{\mu_1^{\Delta \xi}}{\sqrt{n}}$$
$$\le O_p\left(\frac{1}{\sqrt{T}}\right) + \frac{M_7}{\sqrt{n}} = O_p\left(\max\left(\frac{1}{\sqrt{T}}, \frac{1}{\sqrt{n}}\right)\right), \tag{C10}$$

which holds for all $i = 1, \dots, n$ since Lemma D3 holds uniformly over all i and j. Moreover, note that for all $i = 1, \dots, n$, it holds that

$$\left\| \frac{\boldsymbol{\epsilon}_i'\boldsymbol{\Gamma}_0^{\Delta \chi}}{\sqrt{n}} \right\| = \sqrt{\frac{1}{n}\sum_{j=1}^n \left(\gamma_{ij}^{\Delta \chi}\right)^2} = \sqrt{\frac{1}{n}\sum_{j=1}^n \left(\boldsymbol{\lambda}_i'\boldsymbol{\Gamma}_0^{\Delta F}\boldsymbol{\lambda}_j\right)^2} \le r^2 C^2, \tag{C11}$$

because of Assumption 2(b) of uniformly bounded loadings, i.e. with C that does not depend on i.

Sample eigenvalues. For the eigenvalues $\mu_j^{\Delta \chi}$ of $\boldsymbol{\Gamma}_0^{\Delta \chi}$ and $\widehat{\mu}_j^{\Delta y}$ of $\widehat{\boldsymbol{\Gamma}}_0^{\Delta y}$, and using Weyl's inequality (C3), we have

$$\left| \frac{\widehat{\mu}_j^{\Delta y}}{n} - \frac{\mu_j^{\Delta \chi}}{n} \right| \le \left\| \frac{\widehat{\boldsymbol{\Gamma}}_0^{\Delta y}}{n} - \frac{\boldsymbol{\Gamma}_0^{\Delta \chi}}{n} \right\| = O_p\left(\max\left(\frac{1}{\sqrt{T}}, \frac{1}{n}\right)\right), \qquad j = 1, \dots, r. \tag{C12}$$

From Lemma D2(i) and (C12), there exists an integer $\bar{n}$, such that for $n > \bar{n}$, we have

$$\frac{\mu_r^{\Delta \chi}}{n} \ge \underline{M}_6, \qquad \frac{\widehat{\mu}_r^{\Delta y}}{n} \ge \underline{M}_6 + O_p\left(\max\left(\frac{1}{\sqrt{T}}, \frac{1}{n}\right)\right). \tag{C13}$$

Define as $\mathbf{M}^{\Delta \chi}$ and $\widehat{\mathbf{M}}^{\Delta y}$ the diagonal $r \times r$ matrices with diagonal elements $\mu_j^{\Delta \chi}$ and $\widehat{\mu}_j^{\Delta y}$, respectively. From (C13), the matrix $n^{-1}\mathbf{M}^{\Delta \chi}$ is invertible for $n > \bar{n}$ and the inverse of $n^{-1}\widehat{\mathbf{M}}^{\Delta y}$ exists with probability tending to one as $n, T \to \infty$. Moreover, by Lemma D2(i), (C12), and (C13), for $n > \bar{n}$ we have

$$\left\| \left(\frac{\mathbf{M}^{\Delta\chi}}{n} \right)^{-1} \right\| = \frac{n}{\mu_r^{\Delta\chi}} \leq \frac{1}{\underline{M}_6}, \tag{C14}$$

which implies $\|(n^{-1}\mathbf{M}^{\Delta\chi})^{-1}\| = O_p(1)$. Then, from (C12) and (C13), we have

$$\begin{aligned} \left\| \left(\frac{\widehat{\mathbf{M}}^{\Delta y}}{n} \right)^{-1} - \left(\frac{\mathbf{M}^{\Delta\chi}}{n} \right)^{-1} \right\| &\leq \left\| \left(\frac{\widehat{\mathbf{M}}^{\Delta y}}{n} \right)^{-1} - \left(\frac{\mathbf{M}^{\Delta\chi}}{n} \right)^{-1} \right\|_F = \sqrt{\sum_{j=1}^r \left(\frac{n}{\widehat{\mu}_j^{\Delta y}} - \frac{n}{\mu_j^{\Delta\chi}} \right)^2} \\ \leq \sum_{j=1}^r n \left| \frac{\widehat{\mu}_j^{\Delta y} - \mu_j^{\Delta\chi}}{\widehat{\mu}_j^{\Delta y}\mu_j^{\Delta\chi}} \right| &\leq \frac{r \max_{j=1,\dots,r} |\widehat{\mu}_j^{\Delta y} - \mu_j^{\Delta\chi}|}{n\underline{M}_6^2 + O_p\left(\max\left(\frac{n}{\sqrt{T}}, 1\right)\right)} = O_p\left(\max\left(\frac{1}{\sqrt{T}}, \frac{1}{n}\right)\right). \end{aligned} \tag{C15}$$

Last, from the identification constraint (C8), we have that $\mathbf{\Gamma}_0^{\Delta F}$ is diagonal with entries $\mathsf{E}(\Delta F_{jt}^2) = \mu_j^{\Delta\chi}/n$ for $j = 1, \dots, r$, which are finite and bounded away from zero because of Lemma D2(i). Then, by Assumption 1(d) $\mathbf{\Gamma}_0^{\Delta\chi}$ has r non-zero distinct eigenvalues. Moreover, (C8) implies also that $n^{-1}\mathbf{\Lambda}'\mathbf{\Lambda} = \mathbf{I}_r$, for any $n \in \mathbb{N}$. Therefore, under our identification constraints, Lemma D2(i) and thus (C13) and (C14) hold for any $n \in \mathbb{N}$. As a consequence, from Lemma D2(i) there exist positive reals $\underline{C}_j, \overline{C}_j$, such that $\underline{C}_j > \overline{C}_{j+1}$ for $j = 1, \dots, r-1$, and, for any $n \in \mathbb{N}$, we have

$$\underline{C}_j \leq \frac{\mu_j^{\Delta\chi}}{n} \leq \overline{C}_j, \qquad j = 1, \dots, r. \tag{C16}$$

Notice that then $\overline{C}_1 \equiv \overline{M}_6$ and $\underline{C}_r \equiv \underline{M}_6$, where $\overline{M}_6$ and $\underline{M}_6$ are defined in Lemma D2(i).

Sample eigenvectors. Define as $\mathbf{w}_j^{\Delta\chi}$ and $\widehat{\mathbf{w}}_j^{\Delta y}$ the $n \times 1$ normalised eigenvectors corresponding to the j-th largest eigenvalue of $\mathbf{\Gamma}_0^{\Delta\chi}$ and $\widehat{\mathbf{\Gamma}}_0^{\Delta y}$, respectively. Define $s_j = \text{sign}(\widehat{\mathbf{w}}_j^{\Delta y\prime}\mathbf{w}_j^{\Delta\chi})$ and notice that $\widehat{\mathbf{w}}_j^{\Delta y\prime}\mathbf{w}_j^{\Delta\chi}s_j \geq 0$ for all $j = 1, \dots, r$. Then, from Corollary 1 in Yu et al. (2015), defining $\mu_0^{\Delta\chi} = \infty$, we have

$$\|\widehat{\mathbf{w}}_j^{\Delta y} - \mathbf{w}_j^{\Delta\chi}s_j\| \leq \frac{2^{3/2}\|\widehat{\mathbf{\Gamma}}_0^{\Delta y} - \mathbf{\Gamma}_0^{\Delta\chi}\|}{\min\left((\mu_{j-1}^{\Delta\chi} - \mu_j^{\Delta\chi}), (\mu_j^{\Delta\chi} - \mu_{j+1}^{\Delta\chi})\right)}, \qquad j = 1, \dots, r. \tag{C17}$$

Then, because of (C16) for the denominator of (C17), for any $n \in \mathbb{N}$ we have

$$\mu_{j-1}^{\Delta\chi} - \mu_j^{\Delta\chi} \geq n(\underline{C}_{j-1} - \overline{C}_j) > 0, \quad j = 2, \dots, r, \tag{C18}$$

$$\mu_j^{\Delta\chi} - \mu_{j+1}^{\Delta\chi} \geq n(\underline{C}_j - \overline{C}_{j+1}) > 0, \quad j = 1, \dots, r. \tag{C19}$$

Define $\mathbf{J}$ as the $r \times r$ diagonal matrix with entries s_j and define also the $n \times r$ orthonormal matrices of eigenvectors $\mathbf{W}^{\Delta\chi} = (\mathbf{w}_1^{\Delta\chi} \cdots \mathbf{w}_r^{\Delta\chi})$ and $\widehat{\mathbf{W}}^{\Delta y} = (\widehat{\mathbf{w}}_1^{\Delta y} \cdots \widehat{\mathbf{w}}_r^{\Delta y})$. Then, from (C17), (C18), and (C19), we have

$$\|\widehat{\mathbf{W}}^{\Delta y} - \mathbf{W}^{\Delta\chi}\mathbf{J}\| \leq \sqrt{\sum_{j=1}^r \|\widehat{\mathbf{w}}_j^{\Delta y} - \mathbf{w}_j^{\Delta\chi}s_j\|^2} = O_p\left(\max\left(\frac{1}{\sqrt{T}}, \frac{1}{n}\right)\right). \tag{C20}$$

We can now prove part (i). The loadings estimator is defined as $\widehat{\mathbf{\Lambda}} = n^{1/2}\widehat{\mathbf{W}}^{\Delta y}$ while from (C8) we have $\mathbf{\Lambda} = n^{1/2}\mathbf{W}^{\Delta\chi}$. Hence, $\widehat{\boldsymbol{\lambda}}_i' = n^{1/2}\boldsymbol{\epsilon}_i'\widehat{\mathbf{W}}^{\Delta y}$ and $\boldsymbol{\lambda}_i' = n^{1/2}\boldsymbol{\epsilon}_i'\mathbf{W}^{\Delta\chi}$. Then, notice that the columns of $\mathbf{W}^{\Delta\chi}\mathbf{J}$ are also normalised eigenvectors of $\mathbf{\Gamma}_0^{\Delta\chi}$, that is $\mathbf{\Gamma}_0^{\Delta\chi}\mathbf{W}^{\Delta\chi}\mathbf{J} = \mathbf{W}^{\Delta\chi}\mathbf{J}\mathbf{M}^{\Delta\chi}$. Therefore, using

(C10), (C11), (C14), (C15), and (C20), for all $i = 1, \dots, n$ we have

$$
\begin{aligned}
&\|\widehat{\boldsymbol{\lambda}}_i' - \boldsymbol{\lambda}_i'\mathbf{J}\| = \|\sqrt{n}\boldsymbol{\epsilon}_i'\widehat{\mathbf{W}}^{\Delta y} - \sqrt{n}\boldsymbol{\epsilon}_i'\mathbf{W}^{\Delta\chi}\mathbf{J}\| = \left\| \frac{\boldsymbol{\epsilon}_i'}{\sqrt{n}} \left[\widehat{\boldsymbol{\Gamma}}_0^{\Delta y}\widehat{\mathbf{W}}^{\Delta y}\left(\frac{\widehat{\mathbf{M}}^{\Delta y}}{n}\right)^{-1} - \boldsymbol{\Gamma}_0^{\Delta\chi}\mathbf{W}^{\Delta\chi}\mathbf{J}\left(\frac{\mathbf{M}^{\Delta\chi}}{n}\right)^{-1}\right]\right\| \\
&\quad \le \left\|\frac{\boldsymbol{\epsilon}_i'}{\sqrt{n}}(\widehat{\boldsymbol{\Gamma}}_0^{\Delta y} - \boldsymbol{\Gamma}_0^{\Delta\chi})\right\| \left\|\left(\frac{\mathbf{M}^{\Delta\chi}}{n}\right)^{-1}\right\| + \left\|\frac{\boldsymbol{\epsilon}_i'\boldsymbol{\Gamma}_0^{\Delta\chi}}{\sqrt{n}}\right\| \left\|\left(\frac{\widehat{\mathbf{M}}^{\Delta y}}{n}\right)^{-1} - \left(\frac{\mathbf{M}^{\Delta\chi}}{n}\right)^{-1}\right\| \qquad \text{(C21)} \\
&\quad + \|\widehat{\mathbf{W}}^{\Delta y} - \mathbf{W}^{\Delta\chi}\mathbf{J}\| \left\|\frac{\boldsymbol{\epsilon}_i'\boldsymbol{\Gamma}_0^{\Delta\chi}}{\sqrt{n}}\right\| \left\|\left(\frac{\mathbf{M}^{\Delta\chi}}{n}\right)^{-1}\right\| + o_p\left(\max\left(\frac{1}{\sqrt{T}}, \frac{1}{\sqrt{n}}\right)\right) = O_p\left(\max\left(\frac{1}{\sqrt{T}}, \frac{1}{\sqrt{n}}\right)\right),
\end{aligned}
$$

where we also used the fact that $\|\mathbf{W}^{\Delta\chi}\| = 1$. Note in particular that (C21) holds uniformly over all i because of (C10) and (C11)) This proves part (i).

Turning to part (ii), for any $i \in \mathcal{I}_b$, consider $\widehat{b}_i$ defined in (17), then because of (13),

$$
\mathsf{E}[|\widehat{b}_i - b_i|^2] = \mathsf{E}\left[\left(\frac{\sum_{t=1}^T (t - \frac{T+1}{2})(x_{it} - \bar{x}_i)}{\sum_{t=1}^T (t - \frac{T+1}{2})^2}\right)^2\right] = \frac{\mathsf{E}\left[\left(\sum_{t=1}^T t x_{it} - \frac{T+1}{2}\sum_{t=1}^T x_{it}\right)^2\right]}{\left(\frac{1}{12}T(T^2-1)\right)^2}, \qquad \text{(C22)}
$$

where $\bar{y}_i = T^{-1}\sum_{t=1}^T y_{it}$ and $\bar{x}_i = T^{-1}\sum_{t=1}^T x_{it}$ and therefore $\bar{y}_i = \bar{x}_i + a_i + b_i(T+1)/2$. Then, for all $i \in \mathcal{I}_b$, we have

$$
\begin{aligned}
&\mathsf{E}\left[\left(\sum_{t=1}^T x_{it}\right)^2\right] \le 2\left\{\mathsf{E}\left[\left(\sum_{t=1}^T \boldsymbol{\lambda}_i'\mathbf{F}_t\right)^2\right] + \mathsf{E}\left[\left(\sum_{t=1}^T \xi_{it}\right)^2\right]\right\} \le 2C^2\mathsf{E}\left[\left\|\sum_{t=1}^T \mathbf{F}_t\right\|^2\right] + 2\mathsf{E}\left[\left(\sum_{t=1}^T \xi_{it}\right)^2\right] \\
&\quad \le 2C^2 \sum_{t=1}^T\sum_{s=1}^T \left\{\sum_{j_1,j_2=1}^r \left|\mathsf{E}[F_{j_1t}F_{j_2s}]\right| + \left|\mathsf{E}[\xi_{it}\xi_{is}]\right|\right\} \le 2C^2T^2\left(r\mathsf{E}[\|\mathbf{F}_t\|^2] + \mathsf{E}[\xi_{it}^2]\right) = O(T^3), \qquad \text{(C23)}
\end{aligned}
$$

because of Assumption 2(b) of uniformly bounded loadings and Lemma D4(ii) and D4(iv) (and specifically since $\mathsf{E}[\xi_{it}^2] = O(T)$ holds uniformly over i, see also (D13)) and using Cauchy-Schwarz inequality. Moreover, by the same arguments leading to (C23), we also have

$$
\begin{aligned}
&\mathsf{E}\left[\left(\sum_{t=1}^T t x_{it}\right)^2\right] \le 2\left\{\mathsf{E}\left[\left(\sum_{t=1}^T t\boldsymbol{\lambda}_i'\mathbf{F}_t\right)^2\right] + \mathsf{E}\left[\left(\sum_{t=1}^T t\xi_{it}\right)^2\right]\right\} \le 2C^2\mathsf{E}\left[\left\|\sum_{t=1}^T t\mathbf{F}_t\right\|^2\right] + 2\mathsf{E}\left[\left(\sum_{t=1}^T t\xi_{it}\right)^2\right] \\
&\quad \le 4C^2 \sum_{t=1}^T\sum_{s=1}^t ts\left\{\sum_{j_1,j_2=1}^r \left|\mathsf{E}[F_{j_1t}F_{j_2s}]\right| + \left|\mathsf{E}[\xi_{it}\xi_{is}]\right|\right\} \le 4C^2\sum_{t=1}^T \frac{t^2(t+1)}{2}\left(r\mathsf{E}[\|\mathbf{F}_t\|^2] + \mathsf{E}[\xi_{it}^2]\right) \\
&\quad = 4C^2\frac{T(T+1)(T+2)(3T+1)}{24}\left(r\mathsf{E}[\|\mathbf{F}_t\|^2] + \mathsf{E}[\xi_{it}^2]\right) = O(T^5). \qquad \text{(C24)}
\end{aligned}
$$

From (C23) and (C24) we have that the numerator in (C22) is $O(T^5)$. Therefore, $\mathsf{E}[|\widehat{b}_i - b_i|^2] = O(T^{-1})$, for all $i \in \mathcal{I}_b$ and by Chebychev's inequality we prove part (ii).

We can now prove part (iii). First, note that by substituting the expressions for $\boldsymbol{\Lambda}$ and $\widehat{\boldsymbol{\Lambda}}$ in (C20), we have

$$
\left\|\frac{\widehat{\boldsymbol{\Lambda}} - \boldsymbol{\Lambda}\mathbf{J}}{\sqrt{n}}\right\| = \|\widehat{\mathbf{W}}^{\Delta x} - \mathbf{W}^{\Delta\chi}\mathbf{J}\| = O_p\left(\max\left(\frac{1}{\sqrt{T}}, \frac{1}{n}\right)\right), \qquad \text{(C25)}
$$

which implies also that

$$
\left\|\frac{\widehat{\boldsymbol{\Lambda}}'\boldsymbol{\Lambda}}{n} - \mathbf{J}\right\| = O_p\left(\max\left(\frac{1}{\sqrt{T}}, \frac{1}{n}\right)\right). \qquad \text{(C26)}
$$

Then, let $\widehat{\mathbf{b}} = (\widehat{b}_1 \cdots \widehat{b}_n)'$, where $\widehat{b}_i$ is given in (17) if $i \in \mathcal{I}_b$, while $\widehat{b}_i = 0$ otherwise and define the de-trended data as $\widehat{\mathbf{x}}_t = \mathbf{y}_t - \widehat{\mathbf{b}}t$. The factors are estimated as $\widehat{\mathbf{F}}_t = n^{-1}\widehat{\boldsymbol{\Lambda}}'\widehat{\mathbf{x}}_t$. Let also $\mathbf{b} = (b_1 \cdots b_n)'$

and $\mathbf{a} = (a_1 \cdots a_n)'$ such that $\mathbf{y}_t = \mathbf{a} + \mathbf{b}t + \mathbf{x}_t$. Then, for a given t we have

$$\frac{1}{\sqrt{T}}\|\widehat{\mathbf{F}}_t - \mathbf{J}\mathbf{F}_t\| = \left\|\frac{\widehat{\mathbf{\Lambda}}'\widehat{\mathbf{x}}_t}{n\sqrt{T}} - \frac{\mathbf{J}\mathbf{F}_t}{\sqrt{T}}\right\| \le \left\|\frac{\widehat{\mathbf{\Lambda}}'\mathbf{\Lambda}\mathbf{F}_t}{n\sqrt{T}} - \frac{\mathbf{J}\mathbf{F}_t}{\sqrt{T}} + \frac{\widehat{\mathbf{\Lambda}}'\boldsymbol{\xi}_t}{n\sqrt{T}}\right\| + \left\|\frac{\widehat{\mathbf{\Lambda}}'(\mathbf{b}-\widehat{\mathbf{b}})t}{n\sqrt{T}}\right\| + \left\|\frac{\widehat{\mathbf{\Lambda}}'\mathbf{a}}{n\sqrt{T}}\right\|. \quad \text{(C27)}$$

The first term on the rhs of (C27), is such that

$$\begin{aligned}\left\|\frac{\widehat{\mathbf{\Lambda}}'\mathbf{\Lambda}\mathbf{F}_t}{n\sqrt{T}} - \frac{\mathbf{J}\mathbf{F}_t}{\sqrt{T}} + \frac{\widehat{\mathbf{\Lambda}}'\boldsymbol{\xi}_t}{n\sqrt{T}}\right\| &\le \left\|\frac{\widehat{\mathbf{\Lambda}}'\mathbf{\Lambda}}{n} - \mathbf{J}\right\| \left\|\frac{\mathbf{F}_t}{\sqrt{T}}\right\| + \left\|\frac{\widehat{\mathbf{\Lambda}} - \mathbf{\Lambda}\mathbf{J}}{\sqrt{n}}\right\| \left\|\frac{\boldsymbol{\xi}_t}{\sqrt{nT}}\right\| + \left\|\frac{\mathbf{\Lambda}'\boldsymbol{\xi}_t}{n\sqrt{T}}\right\| \|\mathbf{J}\| \\ &= O_p\left(\max\left(\frac{1}{\sqrt{T}}, \frac{1}{n}\right)\right) + O_p\left(\frac{1}{\sqrt{n}}\right), \end{aligned} \quad \text{(C28)}$$

because of (C26), (C25), and Lemma D4(ii), D4(iv) and D4(vi) and since obviously $\|\mathbf{J}\| = 1$.

The second term on the rhs of (C27) is such that

$$\left\|\frac{\widehat{\mathbf{\Lambda}}'(\mathbf{b}-\widehat{\mathbf{b}})t}{n\sqrt{T}}\right\| \le \left\|\frac{\widehat{\mathbf{\Lambda}} - \mathbf{\Lambda}\mathbf{J}}{\sqrt{n}}\right\| \left\|\frac{(\mathbf{b}-\widehat{\mathbf{b}})t}{\sqrt{nT}}\right\| + \left\|\frac{\mathbf{\Lambda}'(\mathbf{b}-\widehat{\mathbf{b}})t}{n\sqrt{T}}\right\| \|\mathbf{J}\|. \quad \text{(C29)}$$

Now, because of part (ii), we have

$$\mathsf{E}\left[\left\|\frac{(\mathbf{b}-\widehat{\mathbf{b}})t}{\sqrt{nT}}\right\|^2\right] = \frac{t^2}{nT}\sum_{i\in\mathcal{I}_b}\mathsf{E}\big[(b_i - \widehat{b}_i)^2\big] = O\left(\frac{1}{n^{1-\eta}}\right). \quad \text{(C30)}$$

since $t \le T$ and by (C25) the first term on the rhs of (C29) is $o_p(\max(T^{-1/2}, n^{-1}))$. For the second term on the rhs of (C29) we have (obviously $\|\mathbf{J}\|^2 = 1$)

$$\begin{aligned}\mathsf{E}\left[\left\|\frac{\mathbf{\Lambda}'(\mathbf{b}-\widehat{\mathbf{b}})t}{n\sqrt{T}}\right\|^2\right] &\le \frac{t^2}{n^2T}\sum_{j=1}^r \mathsf{E}\left[\left(\sum_{i\in\mathcal{I}_b}\lambda_{ij}(b_i - \widehat{b}_i)\right)^2\right] \le \frac{t^2C^2}{n^2T}\sum_{i\in\mathcal{I}_b}\sum_{j\in\mathcal{I}_b}\big|\mathsf{E}[(b_i - \widehat{b}_i)(b_j - \widehat{b}_j)]\big| \\ &\le \frac{TC^2n^\eta}{n^2}\sum_{i\in\mathcal{I}_b}\mathsf{E}[(b_i - \widehat{b}_i)^2] = O\left(\frac{1}{n^{2(1-\eta)}}\right), \end{aligned} \quad \text{(C31)}$$

where we used Assumption 2(b) of uniformly bounded loadings, Cauchy-Schwarz inequality and part (ii). Therefore, (C29) is $O_p(n^{-(1-\eta)})$.

For the third term on the rhs of (C27), since $\|\mathbf{a}\| = O(\sqrt{n})$, we have

$$\left\|\frac{\widehat{\mathbf{\Lambda}}'\mathbf{a}}{n\sqrt{T}}\right\| \le \left\|\frac{\widehat{\mathbf{\Lambda}} - \mathbf{\Lambda}\mathbf{J}}{\sqrt{n}}\right\| \left\|\frac{\mathbf{a}}{\sqrt{nT}}\right\| + \left\|\frac{\mathbf{\Lambda}'\mathbf{a}}{n\sqrt{T}}\right\| \|\mathbf{J}\| = O_p\left(\frac{1}{\sqrt{T}}\right), \quad \text{(C32)}$$

By substituting (C28), (C29), and (C32) into (C27) we prove part (iii). This completes the proof. □

D Auxiliary Lemmas

Lemma D1 *Under Assumptions 1 through 3, there exists a positive real M_5 such that $\mu_1^\varepsilon \le M_5$ and $n^{-1}\sum_{i=1}^n\sum_{j=1}^n |\mathsf{E}[\varepsilon_{it}\varepsilon_{jt}]| \le M_5$, for any $n\in\mathbb{N}$.*

Proof. First notice that, from Assumption 3(b), we have

$$\frac{1}{n}\sum_{i,j=1}^n |\mathsf{E}[\varepsilon_{it}\varepsilon_{jt}]| \le \max_{i=1,\dots,n}\sum_{j=1}^n |\mathsf{E}[\varepsilon_{it}\varepsilon_{jt}]| = \|\mathbf{\Gamma}_0^\varepsilon\|_1 \le M_3.$$

Thus, from (C2), we have $\mu_1^\varepsilon = \left\|\mathbf{\Gamma}_0^\varepsilon\right\| \le \left\|\mathbf{\Gamma}_0^\varepsilon\right\|_1 \le M_3$. By setting $M_5 = M_3$, we complete the proof. □

Lemma D2 *Under Assumptions 1 through 3, there exist positive reals $\underline{M}_6$, $\overline{M}_6$, M_7, $\underline{M}_8$, $\overline{M}_8$ and an integer $\bar{n}$ such that*

(i) $\underline{M}_6 \le n^{-1}\mu_j^{\Delta\chi} \le \overline{M}_6$ for any $j=1,\dots,r$ and $n>\bar{n}$;

(ii) $\mu_1^{\Delta\xi} \le M_7$, for any $n\in\mathbb{N}$;

(iii) $\underline{M}_8 \le n^{-1}\mu_j^{\Delta x} \le \overline{M}_8$ for any $j=1,\dots,r$ and $n>\bar{n}$;

(iv) $\mu_{r+1}^{\Delta x} \le M_7$, for any $n\in\mathbb{N}$.

Proof. Throughout, let $\mathbf{\Gamma}_0^{\Delta F} = \mathsf{E}[\Delta\mathbf{F}_t\Delta\mathbf{F}_t']$, $\mathbf{\Gamma}_0^{\Delta\chi} = \mathsf{E}[\Delta\boldsymbol{\chi}_t\Delta\boldsymbol{\chi}_t']$, $\mathbf{\Gamma}_0^{\Delta\xi} = \mathsf{E}[\Delta\boldsymbol{\xi}_t\Delta\boldsymbol{\xi}_t']$, and $\mathbf{\Gamma}_0^{\Delta x} = \mathsf{E}[\Delta\mathbf{x}_t\Delta\mathbf{x}_t']$. Then, we can write $\mathbf{\Gamma}_0^{\Delta F} = \mathbf{W}^{\Delta F}\mathbf{M}^{\Delta F}\mathbf{W}^{\Delta F'}$, where $\mathbf{W}^{\Delta F}$ is the $r\times r$ matrix of normalised eigenvectors and $\mathbf{M}^{\Delta F}$ the corresponding diagonal matrix of eigenvalues. Define a new $n\times r$ loadings matrix $\boldsymbol{L} = \mathbf{\Lambda}\mathbf{W}^{\Delta F}(\mathbf{M}^{\Delta F})^{1/2}$. Under Assumption 2(a) there exists an integer $\bar{n}$ such that $n^{-1}\mathbf{\Lambda}'\mathbf{\Lambda} = \mathbf{I}_r$, for any $n>\bar{n}$, therefore, for any $n\ge\bar{n}$,

$$\frac{\boldsymbol{L}'\boldsymbol{L}}{n} = \mathbf{M}^{\Delta F}. \tag{D1}$$

By Assumption 1(d) and square summability of the coefficients given in (C5), all eigenvalues of $\mathbf{\Gamma}_0^{\Delta F}$ are positive and finite, i.e. there exist positive reals $\underline{M}_6$ and $\overline{M}_6$ such that

$$\underline{M}_6 \le \mu_j^{\Delta F} \le \overline{M}_6, \quad j=1,\dots,r. \tag{D2}$$

Then, for $n>\bar{n}$,

$$\frac{\mathbf{\Gamma}_0^{\Delta\chi}}{n} = \frac{\mathbf{\Lambda}\mathbf{W}^{\Delta F}\mathbf{M}^{\Delta F}\mathbf{W}^{\Delta F'}\mathbf{\Lambda}'}{n} = \frac{\boldsymbol{L}\boldsymbol{L}'}{n}.$$

Therefore, the non-zero eigenvalues of $\mathbf{\Gamma}_0^{\Delta\chi}$ are the same as those of $\boldsymbol{L}'\boldsymbol{L}$, and from (D1), we have $n^{-1}\mu_j^{\Delta\chi} = \mu_j^{\Delta F}$, for any $n>\bar{n}$ and any $j=1,\dots,r$. Part (i) then follows from (D2).

As for part (ii), we have

$$\mu_1^{\Delta\xi} = \left\|\mathbf{\Gamma}_0^{\Delta\xi}\right\| \le \sum_{k=0}^\infty \left\|\check{\mathbf{D}}_k\right\|^2 \left\|\mathbf{\Gamma}_0^\varepsilon\right\| \le K_2 M_3 = M_7, \tag{D3}$$

because of square summability of the coefficients, with K_2 defined in (C7), and from Lemma D1.

Finally, parts (iii) and (iv) are immediate consequences of Assumption 3(e) of independent common and idiosyncratic shocks, which implies that $\mathbf{\Gamma}_0^{\Delta x} = \mathbf{\Gamma}_0^{\Delta\chi} + \mathbf{\Gamma}_0^{\Delta\xi}$ and of Weyl's inequality (C3). So, because of parts (i) and (ii), there exist positive reals $\underline{M}_8$ and $\overline{M}_8$, such that, for $j=1,\dots,r$, and for any $n>\bar{n}$,

$$\frac{\mu_j^{\Delta x}}{n} \le \frac{\mu_j^{\Delta\chi}}{n} + \frac{\mu_1^{\Delta\xi}}{n} \le \overline{M}_6 + \frac{\mu_1^{\Delta\xi}}{n} \le \overline{M}_6 + \frac{M_7}{n} = \overline{M}_8, \quad \frac{\mu_j^{\Delta x}}{n} \ge \frac{\mu_j^{\Delta\chi}}{n} + \frac{\mu_n^{\Delta\xi}}{n} \ge \underline{M}_6 + \frac{\mu_n^{\Delta\xi}}{n} = \underline{M}_8,$$

This proves part (iii). When $j=r+1$, using parts (i) and (ii), and since $\mathrm{rk}(\mathbf{\Gamma}_0^{\Delta\chi}) = r$, we have $\mu_{r+1}^{\Delta x} \le \mu_{r+1}^{\Delta\chi} + \mu_1^{\Delta\xi} = \mu_1^{\Delta\xi} \le M_7$, thus proving part (iv). This completes the proof. □

Lemma D3 *Let the generic (i,j)-th element of the covariance matrix $\mathbf{\Gamma}_0^{\Delta x}$ of $\Delta\mathbf{x}_t$ be $\gamma_{ij}^{\Delta x} = \mathsf{E}[\Delta x_{it}\Delta x_{jt}]$. Let the generic (i,j)-th element of the sample covariance matrix $\widehat{\mathbf{\Gamma}}_0^{\Delta y}$ of $\Delta\mathbf{y}_t$ be $\widehat{\gamma}_{ij}^{\Delta y}$. Then, under Assumptions 1 through 4, as $T\to\infty$, there exists a positive real C_0 which does not depend on i and j such that $\mathsf{E}[|\widehat{\gamma}_{ij}^{\Delta y} - \gamma_{ij}^{\Delta x}|^2] \le C_0 T^{-1}$.*

Proof. First, note that $\gamma_{ij}^{\Delta x} = \boldsymbol{\lambda}_i'\mathbf{\Gamma}_0^{\Delta F}\boldsymbol{\lambda}_j + \gamma_{ij}^{\Delta\xi}$, where $\boldsymbol{\lambda}_i'$ is the i-th row of $\mathbf{\Lambda}$, $\mathbf{\Gamma}_0^{\Delta F} = \mathsf{E}[\Delta\mathbf{F}_t\Delta\mathbf{F}_t']$, and $\gamma_{ij}^{\Delta\xi} = \mathsf{E}[\Delta\xi_{it}\Delta\xi_{jt}]$.

Start with the sample covariance of the factors, and consider the fourth moments of $\Delta\mathbf{F}_t$. Using (C4), we have

$$
\begin{aligned}
\sum_{t,s=1}^T \mathsf{E}\big[\Delta F_{it}\Delta F_{jt}\Delta F_{is}\Delta F_{js}\big] &= \sum_{t,s=1}^T \sum_{l,l',h,h'=1}^q \sum_{k,k',m,m'=0}^\infty \mathsf{E}\big[c_{ilk}u_{lt-k}c_{il'k'}u_{l't-k'}c_{jhm}u_{hs-m}c_{jh'm'}u_{h's-m'}\big]\\
&\le q^4K_1^4\sum_{t,s=1}^T \mathsf{E}[u_{lt}u_{l't}u_{hs}u_{h's}] = q^4K_1^4\bigg(\sum_{t,s=1}^T \mathsf{E}[u_{lt}^2]\mathsf{E}[u_{hs}^2] + \sum_{t=1}^T \mathsf{E}[u_{lt}^2u_{ht}^2] + \sum_{t=1}^T \mathsf{E}[u_{lt}^4]\bigg), \qquad \text{(D4)}
\end{aligned}
$$

because of Assumption 1(a) of independence of $\mathbf{u}_t$ and square summability of the coefficients, with K_1 defined in (C5). Similarly, for any (i,j)-th element of $\mathbf{\Gamma}_0^{\Delta F}$, denoted as $\gamma_{ij}^{\Delta F}$, we have

$$
\begin{aligned}
(\gamma_{ij}^{\Delta F})^2 = \big(\mathsf{E}\big[\Delta F_{it}\Delta F_{jt}\big]\big)^2 &= \bigg(\sum_{l,l'=1}^q \sum_{k,k'=0}^\infty \mathsf{E}\big[c_{ilk}u_{lt-k}c_{il'k'}u_{l't-k'}\big]\bigg)^2\\
&\le q^4K_1^4 \sum_{t,s=1}^T (\mathsf{E}[u_{lt}u_{l't}]\mathsf{E}[u_{hs}u_{h's}]) = q^4K_1^4\bigg(\sum_{t,s=1}^T \mathsf{E}[u_{lt}^2]\mathsf{E}[u_{hs}^2] + \sum_{t=1}^T (\mathsf{E}[u_{lt}^2])^2\bigg). \qquad \text{(D5)}
\end{aligned}
$$

Now, using (C2) and combining (D4) and (D5), we have

$$
\begin{aligned}
\mathsf{E}\bigg[\bigg\|\frac{1}{T}\sum_{t=1}^T \Delta\mathbf{F}_t\Delta\mathbf{F}_t' - \mathbf{\Gamma}_0^{\Delta F}\bigg\|^2\bigg] &\le \sum_{i,j=1}^r \frac{1}{T^2}\mathsf{E}\bigg[\sum_{t,s=1}^T \Big(\Delta F_{it}\Delta F_{jt} - \gamma_{ij}^{\Delta F}\Big)\Big(\Delta F_{is}\Delta F_{js} - \gamma_{ij}^{\Delta F}\Big)\bigg]\\
&= \sum_{i,j=1}^r \frac{1}{T^2}\sum_{t,s=1}^T \Big(\mathsf{E}\big[\Delta F_{it}\Delta F_{jt}\Delta F_{is}\Delta F_{js}\big] - (\gamma_{ij}^{\Delta F})^2\Big)\\
&= \frac{r^2K_1^4q^4}{T^2}\sum_{t=1}^T \mathsf{E}[u_{lt}^2]\mathsf{E}[u_{ht}^2] + \frac{r^2K_1^4q^4}{T^2}\sum_{t=1}^T \mathsf{E}[u_{lt}^4] - \frac{r^2K_1^4q^4}{T^2}\sum_{t=1}^T (\mathsf{E}[u_{lt}^2])^2 \le \frac{r^2K_1^4q^4M_1}{T}, \qquad \text{(D6)}
\end{aligned}
$$

since $\mathsf{E}[u_{jt}^2] = 1$ for any $j = 1,\dots,q$ and because of Assumption 1(a) of existence of fourth moments.

In the same way, for the idiosyncratic component, using (C6), for all $i,j = 1,\dots,n$, we have

$$
\begin{aligned}
\mathsf{E}\bigg[\bigg|\frac{1}{T}\sum_{t=1}^T \Delta\xi_{it}\Delta\xi_{jt} - \gamma_{ij}^{\Delta\xi}\bigg|^2\bigg] &\le \frac{1}{T^2}\sum_{t,s=1}^T \Big(\mathsf{E}\big[\Delta\xi_{it}\Delta\xi_{jt}\Delta\xi_{is}\Delta\xi_{js}\big] - (\gamma_{ij}^{\Delta\xi})^2\Big)\\
&\le \frac{K_2^4}{T^2}\sum_{t=1}^T \mathsf{E}[\varepsilon_{it}^2\varepsilon_{jt}^2] \le \frac{K_2^4M_2}{T}, \qquad \text{(D7)}
\end{aligned}
$$

where we used Assumption 3(a) of independence of $\boldsymbol{\varepsilon}_t$ and existence of its fourth moments, and square summability of the coefficients, with K_2 defined in (C7). By combining (D6) and (D7) and Assumption 2(b) of uniformly bounded loadings, as $T\to\infty$, there exists a positive real C_1 which does not depend on i and j such that $\mathsf{E}[|\widehat{\gamma}_{ij}^{\Delta x} - \gamma_{ij}^{\Delta x}|^2] \le C_1T^{-1}$.

Then for all $i, j = 1, \dots, n$, we have

$$\begin{aligned}
\mathsf{E}\big[|\widehat{\gamma}_{ij}^{\Delta y} - \widehat{\gamma}_{ij}^{\Delta x}|^2\big] &= \mathsf{E}\left[\left|\frac{1}{T}\sum_{t=1}^{T}\Big(\big(\Delta y_{it} - \Delta\overline{y}_i\big)\big(\Delta y_{jt} - \Delta\overline{y}_j\big) - \Delta x_{it}\Delta x_{jt}\Big)\right|^2\right] \\
&\le 2\mathsf{E}\left[\left|\frac{1}{T}\sum_{t=1}^{T}\Delta x_{it}\big(b_j - \Delta\overline{y}_j\big)\right|^2\right] + \mathsf{E}\left[\left|\frac{1}{T}\sum_{t=1}^{T}\big(b_i - \Delta\overline{y}_i\big)\big(b_j - \Delta\overline{y}_j\big)\right|^2\right] \\
&\le 2\mathsf{E}\left[\left|\frac{1}{T}\sum_{t=1}^{T}\Delta x_{it}\right|^2\right]\mathsf{E}\big[\big|\big(b_i - \Delta\overline{y}_i\big)\big|^2\big] + \mathsf{E}\big[\big|\big(b_i - \Delta\overline{y}_i\big)\big(b_j - \Delta\overline{y}_j\big)\big|^2\big]. \qquad \text{(D8)}
\end{aligned}$$

Now, by definition of sample mean we have for all $i = 1, \dots, n$

$$\begin{aligned}
&\mathsf{E}\big[\big|b_i - \Delta\overline{y}_i\big|^2\big] = \mathsf{E}\left[\left|\frac{1}{T}\sum_{t=1}^{T}\Delta x_{it}\right|^2\right] = \frac{1}{T^2}\sum_{t,s=1}^{T}\big|\mathsf{E}[\Delta x_{it}\Delta x_{is}]\big| \\
&\le \frac{1}{T^2}\sum_{t,s=1}^{T}\big|\mathsf{E}[\boldsymbol{\lambda}_i'\Delta\mathbf{F}_t\boldsymbol{\lambda}_i'\Delta\mathbf{F}_s]\big| + \frac{1}{T^2}\sum_{t,s=1}^{T}\big|\mathsf{E}[\Delta\xi_{it}\Delta\xi_{is}]\big| \\
&\le \frac{C^2}{T^2}\sum_{t,s=1}^{T}\sum_{j,\ell=1}^{r}\sum_{k,h=0}^{\infty}|c_{jm_1k}|\,|c_{\ell m_2h}|\sum_{m_1,m_2=1}^{q}|\mathsf{E}[u_{m_1t-k}u_{m_2s-h}]| + \frac{1}{T^2}\sum_{t,s=1}^{T}\sum_{k,h=0}^{\infty}|d_{ik}|\,|d_{ih}||\mathsf{E}[\varepsilon_{it-k}\varepsilon_{is-h}]| \\
&\le \frac{C^2r^2qK_1^2}{T}\mathsf{E}[u_{jt}^2] + \frac{K_2^2}{T}\max_{i=1,\dots,n}\mathsf{E}[\varepsilon_{it}^2] = O\left(\frac{1}{T}\right), \qquad \text{(D9)}
\end{aligned}$$

because of Assumption 1(a) of independence of $\mathbf{u}_t$ and square summability of the coefficients, with K_1 defined in (C5) and since $\mathsf{E}[u_{jt}^2] = 1$ for any $j = 1, \dots, q$, and because of Assumption 3(a) of independence of $\boldsymbol{\varepsilon}_t$ and existence of its fourth moments, and square summability of the coefficients, with K_2 defined in (C7) and since $\max_{i=1,\dots,n}\mathsf{E}[\varepsilon_{it}^2]$ is finite by Assumption 3(b). By using (D9) in (D8) we have that as $T \to \infty$, there exists a positive real C_2 which does not depend on i and j such that $\mathsf{E}[|\widehat{\gamma}_{ij}^{\Delta y} - \widehat{\gamma}_{ij}^{\Delta x}|^2] \le C_2T^{-1}$.

Therefore,

$$\mathsf{E}\big[|\widehat{\gamma}_{ij}^{\Delta y} - \gamma_{ij}^{\Delta x}|^2\big] \le \mathsf{E}\big[|\widehat{\gamma}_{ij}^{\Delta y} - \widehat{\gamma}_{ij}^{\Delta x}|^2\big] + \mathsf{E}\big[|\widehat{\gamma}_{ij}^{\Delta x} - \gamma_{ij}^{\Delta x}|^2\big] \le \frac{C_1 + C_2}{T}, \qquad \text{(D10)}$$

by setting $C_0 = C_1 + C_2$ we complete the proof. □

Lemma D4 *Under Assumptions 1 through 4, for any t we have*

(i) $\mathsf{E}[\|\Delta\mathbf{F}_t\|^2] = O(1)$;
(ii) $\mathsf{E}[\|T^{-1/2}\mathbf{F}_t\|^2] = O(1)$;
(iii) $\mathsf{E}[\|n^{-1/2}\Delta\boldsymbol{\xi}_t\|^2] = O(1)$;
(iv) $\mathsf{E}[\|(nT)^{-1/2}\boldsymbol{\xi}_t\|^2] = O(1)$;
(v) $\mathsf{E}[\|n^{-1/2}\boldsymbol{\Lambda}'\Delta\boldsymbol{\xi}_t\|^2] = O(1)$;
(vi) $\mathsf{E}[\|(nT)^{-1/2}\boldsymbol{\Lambda}'\boldsymbol{\xi}_t\|^2] = O(1)$.

Proof. For part (i), just notice that, since by Assumption 1(b) $\Delta F_{jt} \sim I(0)$ for any $i = 1, \dots, r$, then they have finite variance. This proves part (i).

For part (ii), from (C4) we have

$$\begin{aligned}
\mathsf{E}\left[\left\|\frac{\mathbf{F}_t}{\sqrt{T}}\right\|^2\right] &= \frac{1}{T}\sum_{j=1}^{r}\mathsf{E}\big[F_{jt}^2\big] = \frac{1}{T}\sum_{j=1}^{r}\mathsf{E}\left[\left(\sum_{s=1}^{t}\sum_{l=1}^{q}c_{jl}(L)u_{ls}\right)^2\right] \\
&= \frac{1}{T}\sum_{j=1}^{r}\sum_{s,s'=1}^{t}\sum_{l,l'=1}^{q}\sum_{k,k'=0}^{\infty}c_{jlk}c_{jl'k'}\mathsf{E}[u_{ls-k}u_{l's'-k'}] \le \frac{rqK_1t}{T} \le rqK_1, \qquad \text{(D11)}
\end{aligned}$$

since $t \leq T$ and where we used the fact $\mathbf{u}_t$ is a white noise because of Assumption 1(a) and we used square summability of the coefficients, with K_1 defined in (C5). This proves part (ii).

For part (iii), for any $n \in \mathbb{N}$ and from (C6), we have,

$$\begin{aligned}\mathsf{E}\left[\left\|\frac{\Delta\boldsymbol{\xi}_t}{\sqrt{n}}\right\|^2\right] &= \frac{1}{n}\sum_{i=1}^n \mathsf{E}\big[\Delta\xi_{it}^2\big] = \frac{1}{n}\sum_{i=1}^n \mathsf{E}[(\check{d}_i(L)\varepsilon_{it})^2]\\ &=\frac{1}{n}\sum_{i=1}^n\sum_{k,k'=0}^\infty \check{d}_{jk}\check{d}_{ik'}\mathsf{E}[\varepsilon_{it-k}\varepsilon_{it-k'}] \leq K_2 \max_{i=1,\ldots,n}\mathsf{E}[\varepsilon_{it}^2],\end{aligned} \tag{D12}$$

where we used Assumption 3(a) of serially uncorrelated $\boldsymbol{\varepsilon}_t$ and square summability of the coefficients, with K_2 defined in (C7). Also because of the existence of fourth moments in Assumption 3(a) the variance of ε_{it} is finite for any i. This proves part (iii).

Similarly, for part (iv), for any $n \in \mathbb{N}$, we have,

$$\begin{aligned}\mathsf{E}\left[\left\|\frac{\boldsymbol{\xi}_t}{\sqrt{nT}}\right\|^2\right] &= \frac{1}{nT}\sum_{i=1}^n \mathsf{E}\big[\xi_{it}^2\big] = \frac{1}{nT}\sum_{i=1}^n \mathsf{E}\left[\left(\sum_{s=1}^t \check{d}_i(L)\varepsilon_{is}\right)^2\right]\\ &=\frac{1}{nT}\sum_{i=1}^n\sum_{s,s'=1}^t\sum_{k,k'=0}^\infty \check{d}_{ik}\check{d}_{ik'}\mathsf{E}[\varepsilon_{is-k}\varepsilon_{is'-k'}] \leq \frac{K_2 t}{T}\max_{i=1,\ldots n}\mathsf{E}[\varepsilon_{it}^2] \leq K_2 \max_{i=1,\ldots,n}\mathsf{E}[\varepsilon_{it}^2],\end{aligned} \tag{D13}$$

since $t \leq T$ and where we used the same assumptions as in (D12). This proves part (iv).

As for part (v), for any $n \in \mathbb{N}$, we have

$$\begin{aligned}\mathsf{E}\left[\left\|\frac{\boldsymbol{\Lambda}'\Delta\boldsymbol{\xi}_t}{\sqrt{n}}\right\|^2\right] &= \frac{1}{n}\sum_{j=1}^r \mathsf{E}\left[\left(\sum_{i=1}^n \lambda_{ij}\Delta\xi_{it}\right)^2\right] = \frac{1}{n}\sum_{j=1}^r\sum_{i,l=1}^n \mathsf{E}\big[\lambda_{ij}\Delta\xi_{it}\lambda_{lj}\Delta\xi_{lt}\big]\\ &\leq\frac{rC^2}{n}\sum_{i,l=1}^n\sum_{k,k'=0}^\infty \check{d}_{ik}\check{d}_{lk'}\mathsf{E}[\varepsilon_{it-k}\varepsilon_{lt-k'}] \leq \frac{rC^2K_2}{n}\sum_{i,l=1}^n\big|\mathsf{E}[\varepsilon_{it}\varepsilon_{lt}]\big| \leq rC^2K_2M_3,\end{aligned} \tag{D14}$$

where we used the same assumptions as in (D12), Assumption 2(b) of bounded loadings, and Lemma D1. This proves part (v).

Similarly for part (vi), for any $n \in \mathbb{N}$, we have

$$\begin{aligned}\mathsf{E}\left[\left\|\frac{\boldsymbol{\Lambda}'\boldsymbol{\xi}_t}{\sqrt{nT}}\right\|^2\right] &= \frac{1}{nT}\sum_{j=1}^r \mathsf{E}\left[\left(\sum_{i=1}^n \lambda_{ij}\xi_{it}\right)^2\right] = \frac{1}{nT}\sum_{j=1}^r\sum_{i,l=1}^n \mathsf{E}\big[\lambda_{ij}\xi_{it}\lambda_{lj}\xi_{lt}\big]\\ &\leq\frac{rC^2}{nT}\sum_{i,l=1}^n\sum_{s,s'=1}^t\sum_{k,k'=0}^\infty \check{d}_{ik}\check{d}_{lk'}\mathsf{E}[\varepsilon_{is-k}\varepsilon_{ls'-k'}] \leq \frac{rC^2K_2t}{nT}\sum_{i,l=1}^n\big|\mathsf{E}[\varepsilon_{it}\varepsilon_{lt}]\big| \leq rC^2K_2M_3,\end{aligned} \tag{D15}$$

where we used the same assumptions as in (D14). This proves part (vi) and completes the proof. □

Lemma D5 *Under Assumptions 1 and 4:*

(i) $\mathbf{F}_t = \mathbf{C}(1)\sum_{s=1}^t \mathbf{u}_s + \check{\mathbf{C}}(L)\mathbf{u}_t$*, such that* $\check{\mathbf{C}}(L)$ *is an* $r \times q$ *infinite rational polynomial matrix with square summable coefficients; moroever,* $\mathbf{C}(1) = \boldsymbol{\psi}\boldsymbol{\eta}'$*, where* $\boldsymbol{\psi}$ *is* $r \times r - c$*,* $\boldsymbol{\eta}$ *is* $q \times r - c$*,* $rk(\boldsymbol{\psi}) = rk(\boldsymbol{\eta}) = r - c = q - d$ *and* $\boldsymbol{\beta}'\mathbf{C}(1) = \mathbf{0}_{c\times q}$*, where* $\boldsymbol{\beta}$ *is the* $r \times c$ *cointegration matrix;*

(ii) $\mathsf{E}[\|\boldsymbol{\beta}'\mathbf{F}_t\|^2] = O(1)$ *for any* $t = 1, \ldots, T$*.*

Proof. From Lemma 2.1 in Phillips and Solo (1992), the Beveridge-Nelson decomposition of $\mathbf{C}(L)$ in (8) gives

$$\Delta\mathbf{F}_t = \mathbf{C}(1)\mathbf{u}_t + \check{\mathbf{C}}(L)(\mathbf{u}_t - \mathbf{u}_{t-1}),$$

where $\check{\mathbf{C}}(L)=\sum_{k=0}^{\infty}\check{\mathbf{C}}_k L^k$ with $\check{\mathbf{C}}_k=-\sum_{h=k+1}^{\infty}\mathbf{C}_h$ and has square summable coefficients because of (C5). Then,

$$\mathbf{F}_t=\mathbf{C}(1)\sum_{s=1}^{t}\mathbf{u}_s+\boldsymbol{\omega}_t, \tag{D16}$$

where $\boldsymbol{\omega}_t=\check{\mathbf{C}}(L)(\mathbf{u}_t-\mathbf{u}_0)=\check{\mathbf{C}}(L)\mathbf{u}_t$, since $\mathbf{u}_t=\mathbf{0}_q$ when $t\le 0$ by Assumption 4, and $\boldsymbol{\omega}_t\sim I(0)$, because of square summability of the coefficients of $\check{\mathbf{C}}(L)$. Moreover, from Assumption 1(c) of cointegration, we have $\mathbf{C}(1)=\boldsymbol{\psi}\boldsymbol{\eta}'$, where $\boldsymbol{\psi}$ is $r\times r-c$ and $\boldsymbol{\eta}$ is $q\times r-c$. Since $\boldsymbol{\beta}$ is a cointegrating vector for $\mathbf{F}_t$, we must have $\boldsymbol{\beta}'\mathbf{F}_t\sim I(0)$, which from (D16) implies $\boldsymbol{\beta}'\mathbf{C}(1)=\mathbf{0}_{c\times q}$. This proves part (i).

Turning to part (ii), from part (i) and (D16), we have

$$\boldsymbol{\beta}'\mathbf{F}_t=\boldsymbol{\beta}'\boldsymbol{\omega}_t=\boldsymbol{\beta}'\check{\mathbf{C}}(L)\mathbf{u}_t.$$

Define $\widetilde{\mathbf{C}}(L)=\boldsymbol{\beta}'\check{\mathbf{C}}(L)$ and notice that it has square summable coefficients because of square summability of the coefficients of $\check{\mathbf{C}}(L)$, then

$$\begin{aligned}\mathsf{E}\big[\|\boldsymbol{\beta}'\mathbf{F}_t\|^2\big]&=\sum_{j=1}^{r}\mathsf{E}[(\widetilde{\mathbf{c}}_j'(L)\mathbf{u}_t)^2]=\sum_{j=1}^{r}\mathsf{E}\Bigg[\bigg(\sum_{l=1}^{q}\widetilde{c}_{jl}(L)u_{lt}\bigg)^2\Bigg]\\&=\sum_{j=1}^{r}\sum_{l,l'=1}^{q}\sum_{k,k'=0}^{\infty}\widetilde{c}_{jlk}\widetilde{c}_{jl'k'}\mathsf{E}[u_{lt-k}u_{l't-k'}]\le rqK_1,\end{aligned} \tag{D17}$$

where we used the fact $\mathbf{u}_t$ is a white noise because of Assumption 1(a) and we used square summability of the coefficients, with K_1 defined in (C5). This proves part (ii) and completes the proof. □

Lemma D6 *For $k=0,1$, define $\boldsymbol{\Gamma}_k^{\Delta F}=\mathsf{E}[\Delta\mathbf{F}_t\Delta\mathbf{F}_{t-k}']$ and $\boldsymbol{\Gamma}_k^{\omega}=\mathsf{E}[\boldsymbol{\omega}_t\boldsymbol{\omega}_{t-k}']$, where $\boldsymbol{\omega}_t=\check{\mathbf{C}}(L)\mathbf{u}_t$ is defined in* (D16)*. Define also, $\boldsymbol{\Gamma}_L^{\omega}=\boldsymbol{\Gamma}_0^{\omega}+2\sum_{h=1}^{\infty}\boldsymbol{\Gamma}_h^{\omega}$. Denote as $\mathbf{W}_q(\cdot)$ a q-dimensional Brownian motion with covariance $\mathbf{I}_q$ and as $\mathbf{W}_r(\cdot)$ an r-dimensional Brownian motion with covariance $\mathbf{I}_r$. Under Assumptions 1 and 4, as $T\to\infty$,*

(i) $\mathsf{E}[\|T^{-1}\sum_{t=k+1}^{T}\Delta\mathbf{F}_t\Delta\mathbf{F}_{t-k}'-\boldsymbol{\Gamma}_k^{\Delta F}\|^2]=O(T^{-1})$, *for* $k=0,1$;

(ii) $T^{-2}\sum_{t=1}^{T}\mathbf{F}_t\mathbf{F}_t'\overset{d}{\to}\mathbf{C}(1)\big(\int_0^1\mathbf{W}_q(\tau)\mathbf{W}_q'(\tau)\mathrm{d}\tau\big)\mathbf{C}'(1)$;

(iii) $T^{-1}\sum_{t=1}^{T}\mathbf{F}_{t-1}\Delta\mathbf{F}_t'\overset{d}{\to}\mathbf{C}(1)\big(\int_0^1\mathbf{W}_q(\tau)\mathrm{d}\mathbf{W}_q'(\tau)\big)\mathbf{C}'(1)+(\boldsymbol{\Gamma}_1^{\omega}-\boldsymbol{\Gamma}_0^{\omega})$;

(iv) $T^{-1}\sum_{t=1}^{T}\mathbf{F}_t\mathbf{F}_t'\boldsymbol{\beta}\overset{d}{\to}\mathbf{C}(1)\big(\int_0^1\mathbf{W}_q(\tau)\mathrm{d}\mathbf{W}_r'(\tau)\big)(\boldsymbol{\Gamma}_L^{\omega})^{1/2}\boldsymbol{\beta}+\boldsymbol{\Gamma}_0^{\omega}\boldsymbol{\beta}$;

(v) $\mathsf{E}[\|T^{-1}\sum_{t=1}^{T}\boldsymbol{\beta}'\mathbf{F}_t\mathbf{F}_t'\boldsymbol{\beta}-\boldsymbol{\beta}'\boldsymbol{\Gamma}_0^{\omega}\boldsymbol{\beta}\|^2]=\mathsf{E}[\|T^{-1}\sum_{t=1}^{T}\boldsymbol{\beta}'\mathbf{F}_t\mathbf{F}_t'\boldsymbol{\beta}-\mathsf{E}[\boldsymbol{\beta}'\mathbf{F}_t\mathbf{F}_t'\boldsymbol{\beta}]\|^2]=O(T^{-1})$;

(vi) $\mathsf{E}[\|T^{-1}\sum_{t=1}^{T}\Delta\mathbf{F}_t\mathbf{F}_{t-1}'\boldsymbol{\beta}-(\boldsymbol{\Gamma}_1^{\omega}-\boldsymbol{\Gamma}_0^{\omega})\boldsymbol{\beta}\|^2]=\mathsf{E}[\|T^{-1}\sum_{t=1}^{T}\Delta\mathbf{F}_t\mathbf{F}_{t-1}'\boldsymbol{\beta}-\mathsf{E}[\Delta\mathbf{F}_t\mathbf{F}_{t-1}'\boldsymbol{\beta}]\|^2]=O(T^{-1})$.

Proof. For part (i), the case $k=0$ is already proved in (D6) in the proof of Lemma D3. The proof for the case $k=1$, is analogous.

In order to prove the other statements, notice that $\mathrm{rk}(\boldsymbol{\Gamma}_L^{\omega})=r$ because of Assumption 1(d) and define, for $\tau\in[0,1]$,

$$\boldsymbol{\mathcal{X}}_{u,T}(\tau)=\frac{1}{\sqrt{T}}\sum_{s=1}^{\lfloor T\tau\rfloor}\mathbf{u}_s,\qquad \boldsymbol{\mathcal{X}}_{\omega,T}(\tau)=\Big(\boldsymbol{\Gamma}_L^{\omega}\Big)^{-1/2}\frac{1}{\sqrt{T}}\sum_{s=1}^{\lfloor T\tau\rfloor}\boldsymbol{\omega}_s.$$

Then, we can write

$$\sum_{s=1}^{t} \mathbf{u}_s = \sqrt{T}\,\boldsymbol{\mathcal{X}}_{u,T}\left(\frac{t}{T}\right), \tag{D18}$$

$$\mathbf{u}_t = \sqrt{T}\left[\boldsymbol{\mathcal{X}}_{u,T}\left(\frac{t}{T}\right) - \boldsymbol{\mathcal{X}}_{u,T}\left(\frac{t-1}{T}\right)\right], \tag{D19}$$

$$\boldsymbol{\omega}_t = \sqrt{T}\left(\boldsymbol{\Gamma}_L^{\omega}\right)^{1/2}\left[\boldsymbol{\mathcal{X}}_{\omega,T}\left(\frac{t}{T}\right) - \boldsymbol{\mathcal{X}}_{\omega,T}\left(\frac{t-1}{T}\right)\right]. \tag{D20}$$

As proved in Corollary 2.2 in Phillips and Durlauf (1986) (see also Theorem 3.4 in Phillips and Solo, 1992), for any $\tau \in [0,1]$, we have, as $T \to \infty$,

$$\boldsymbol{\mathcal{X}}_{u,T}(\tau) \xrightarrow{d} \mathbf{W}_q(\tau), \qquad \boldsymbol{\mathcal{X}}_{\omega,T}(\tau) \xrightarrow{d} \mathbf{W}_r(\tau), \tag{D21}$$

where $\mathbf{W}_q(\cdot)$ is a q-dimensional Brownian motion with covariance $\mathbf{I}_q$ and $\mathbf{W}_r(\cdot)$ is a q-dimensional Brownian motion with covariance $\mathbf{I}_r$.

For part (ii), from Lemma D5(i), we have

$$\begin{aligned}\frac{1}{T^2}\sum_{t=1}^{T}\mathbf{F}_t\mathbf{F}_t' = {}& \frac{1}{T^2}\sum_{t=1}^{T}\left[\left(\mathbf{C}(1)\sum_{s=1}^{t}\mathbf{u}_s\right)\left(\mathbf{C}(1)\sum_{s=1}^{t}\mathbf{u}_s\right)'\right] \\ &+ \frac{1}{T^2}\sum_{t=1}^{T}\left[\left(\mathbf{C}(1)\sum_{s=1}^{t}\mathbf{u}_s\right)\boldsymbol{\omega}_t' + \boldsymbol{\omega}_t\left(\mathbf{C}(1)\sum_{s=1}^{t}\mathbf{u}_s\right)'\right] + \frac{1}{T^2}\sum_{t=1}^{T}\boldsymbol{\omega}_t\boldsymbol{\omega}_t'.\end{aligned} \tag{D22}$$

For the first term on the rhs of (D22), using (D18) and (D21), we have, as $T \to \infty$,

$$\frac{1}{T^2}\sum_{t=1}^{T}\left[\left(\mathbf{C}(1)\sum_{s=1}^{t}\mathbf{u}_s\right)\left(\mathbf{C}(1)\sum_{s=1}^{t}\mathbf{u}_s\right)'\right] \xrightarrow{d} \mathbf{C}(1)\left(\int_0^1 \mathbf{W}_q(\tau)\mathbf{W}_q'(\tau)\mathrm{d}\tau\right)\mathbf{C}'(1), \tag{D23}$$

which is $O_p(1)$, since it has finite covariance, and has rank $r-c$, since $\mathrm{rk}(\mathbf{C}(1)) = r-c$ because of Assumption 1(c). Then, since $\frac{\mathbf{W}_r(\tau)-\mathbf{W}_r(\tau-\mathrm{d}\tau)}{\mathrm{d}\tau} = \frac{\mathrm{d}\mathbf{W}_r(\tau)}{\mathrm{d}\tau} + O(\mathrm{d}\tau)$, as $\mathrm{d}\tau \to 0$, using (D20) and (D21), we have, as $T \to \infty$,

$$\frac{1}{T}\sum_{t=1}^{T}\left(\mathbf{C}(1)\sum_{s=1}^{t}\mathbf{u}_s\right)\boldsymbol{\omega}_t' \xrightarrow{d} \mathbf{C}(1)\left(\int_0^1 \mathbf{W}_q(\tau)\mathrm{d}\mathbf{W}_r'(\tau)\right)\left(\boldsymbol{\Gamma}_L^{\omega}\right)^{1/2}, \tag{D24}$$

which is $O_p(1)$, since it has finite covariance. Therefore, the second and third term on the rhs of (D22) are $O_p(T^{-1})$. Similarly, the fourth term on the rhs of (D22) is $O_p(T^{-1})$ since $\|\boldsymbol{\Gamma}_0^{\omega}\| = O(1)$ and for $k = 0, 1$, we have

$$\mathsf{E}\left[\left\|\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{\omega}_t\boldsymbol{\omega}_{t-k}' - \boldsymbol{\Gamma}_k^{\omega}\right\|^2\right] = O\left(\frac{1}{T}\right), \tag{D25}$$

by arguments analogous to those used in proving part (i). By substituting (D23), (D24), and (D25) (which implies convergence in probability by Chebychev's inequality) in (D22), and by Slutsky's theorem, we prove part (ii).

For part (iii), from Lemma D5(i), we have

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T}\mathbf{F}_{t-1}\Delta\mathbf{F}_t' = {}& \frac{1}{T}\sum_{t=1}^{T}\left[\left(\sum_{s=1}^{t-1}\mathbf{C}(1)\mathbf{u}_s\right)\left(\mathbf{C}(1)\mathbf{u}_t\right)'\right] + \frac{1}{T}\sum_{t=1}^{T}\left[\left(\sum_{s=1}^{t-1}\mathbf{C}(1)\mathbf{u}_s\right)\Delta\boldsymbol{\omega}_t'\right] \\ &+ \frac{1}{T}\sum_{t=1}^{T}\left[\boldsymbol{\omega}_{t-1}\left(\mathbf{C}(1)\mathbf{u}_t\right)'\right] + \frac{1}{T}\sum_{t=1}^{T}\boldsymbol{\omega}_{t-1}\Delta\boldsymbol{\omega}_t'.\end{aligned} \tag{D26}$$

For the first term on the rhs of (D26), using (D18), (D19), and (D21), we have, as $T \to \infty$,

$$\frac{1}{T}\sum_{t=1}^{T}\left[\left(\sum_{s=1}^{t-1}\mathbf{C}(1)\mathbf{u}_s\right)\left(\mathbf{C}(1)\mathbf{u}_t\right)'\right] \xrightarrow{d} \mathbf{C}(1)\left(\int_0^1 \mathbf{W}_q(\tau)\mathrm{d}\mathbf{W}_q'(\tau)\right)\mathbf{C}'(1), \tag{D27}$$

which is $O_p(1)$, since it has finite covariance, and has rank $r-c$, since $\mathrm{rk}(\mathbf{C}(1)) = r-c$. For the second term on the rhs of (D26), since $\Delta\boldsymbol{\omega}_t = \boldsymbol{\omega}_t - \boldsymbol{\omega}_{t-1}$, by following twice the same steps as those leading to (D24), we have

$$\frac{1}{T}\sum_{t=1}^{T}\left[\left(\sum_{s=1}^{t-1}\mathbf{C}(1)\mathbf{u}_s\right)\Delta\boldsymbol{\omega}_t'\right] \xrightarrow{d} \mathbf{0}_{r\times r}. \tag{D28}$$

For the third term on the rhs of (D26) we have

$$\mathsf{E}\left[\left\|\frac{1}{T}\sum_{t=1}^{T}\left[\boldsymbol{\omega}_{t-1}\left(\mathbf{C}(1)\mathbf{u}_t\right)'\right]\right\|^2\right] = O\left(\frac{1}{T}\right). \tag{D29}$$

by arguments similar to (D25) and the fact that $\mathsf{E}[\boldsymbol{\omega}_{t-1}\mathbf{u}_t'] = \mathbf{0}_{r\times r}$, because of orthonormality of $\mathbf{u}_t$ given in Assumption 1(a). Last, for the fourth term on the rhs of (D26), we can use (D25) to show that

$$\mathsf{E}\left[\left\|\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{\omega}_{t-1}\Delta\boldsymbol{\omega}_t' - \left(\boldsymbol{\Gamma}_1^{\omega} - \boldsymbol{\Gamma}_0^{\omega}\right)\right\|^2\right] = O\left(\frac{1}{T}\right). \tag{D30}$$

By substituting (D27), (D28), (D29) and (D30) (both implying convergence in probability by Chebychev's inequality) in (D26), and by Slutsky's theorem, we prove part (iii).

Turning to part (iv), since $\boldsymbol{\beta}'\mathbf{F}_t = \boldsymbol{\beta}'\boldsymbol{\omega}_t$, from Lemma D5(i), we have

$$\begin{aligned}\frac{1}{T}\sum_{t=1}^{T}\mathbf{F}_t\mathbf{F}_t'\boldsymbol{\beta} &= \mathbf{C}(1)\left[\frac{1}{T}\sum_{t=1}^{T}\left(\sum_{s=1}^{t}\mathbf{u}_s\right)\boldsymbol{\omega}_t'\right]\boldsymbol{\beta} + \left[\frac{1}{T}\sum_{t=1}^{T}\boldsymbol{\omega}_t\boldsymbol{\omega}_t'\right]\boldsymbol{\beta}\\ &\xrightarrow{d} \mathbf{C}(1)\left(\int_0^1 \mathbf{W}_q(\tau)\mathrm{d}\mathbf{W}_r'(\tau)\right)\left(\boldsymbol{\Gamma}_L^{\omega}\right)^{1/2}\boldsymbol{\beta} + \boldsymbol{\Gamma}_0^{\omega}\boldsymbol{\beta}.\end{aligned} \tag{D31}$$

by analogous arguments as those leading to (D24) and using (D25) and Slutsky's theorem. This completes the proof of part (iv).

Part (v) is proved analogously just by multiplying (D31) also on the left by $\boldsymbol{\beta}'$ and then using (D25) and the fact that $\boldsymbol{\beta}'\mathbf{F}_t = \boldsymbol{\beta}'\boldsymbol{\omega}_t$ because of Lemma D5(i).

Finally, part (vi) is proved by noticing that

$$\frac{1}{T}\sum_{t=1}^{T}\Delta\mathbf{F}_t\mathbf{F}_{t-1}'\boldsymbol{\beta} = \left(\frac{1}{T}\sum_{t=1}^{T}\mathbf{C}(1)\mathbf{u}_t\boldsymbol{\omega}_{t-1}' + \frac{1}{T}\sum_{t=1}^{T}\Delta\boldsymbol{\omega}_t\boldsymbol{\omega}_{t-1}'\right)\boldsymbol{\beta}$$

and using (D29) and (D30). This completes the proof. □

Lemma D7 *Under Assumptions 1 through 4 and 6, as $n, T \to \infty$,*

(i) $\mathsf{E}[\|(nT^2)^{-1}\sum_{t=1}^{T}\mathbf{F}_t\boldsymbol{\xi}_t'\boldsymbol{\Lambda}\|^2] = O(n^{-(2-\delta)})$;
(ii) $\mathsf{E}[\|(\sqrt{n}T^2)^{-1}\sum_{t=1}^{T}\mathbf{F}_t\boldsymbol{\xi}_t'\|^2] = O(n^{-(1-\delta)})$;
(iii) $\mathsf{E}[\|(n^2T^2)^{-1}\sum_{t=1}^{T}\boldsymbol{\Lambda}'\boldsymbol{\xi}_t\boldsymbol{\xi}_t'\boldsymbol{\Lambda}\|^2] = O(n^{-2(2-\delta)})$;
(iv) $\mathsf{E}[\|(nT^2)^{-1}\sum_{t=1}^{T}\boldsymbol{\xi}_t\boldsymbol{\xi}_t'\|^2] = O(n^{-2(1-\delta)})$;
(v) $\mathsf{E}[\|(nT)^{-1}\sum_{t=1}^{T}\Delta\mathbf{F}_t\boldsymbol{\xi}_t'\boldsymbol{\Lambda}\|^2] = O(Tn^{-(2-\delta)})$;

(vi) $\mathsf{E}[\|(\sqrt{n}T)^{-1}\sum_{t=1}^T \Delta\mathbf{F}_t\boldsymbol{\xi}_t'\|^2] = O(Tn^{-(1-\delta)})$;

(vii) $\mathsf{E}[\|(n^2T)^{-1}\sum_{t=1}^T \boldsymbol{\Lambda}'\Delta\boldsymbol{\xi}_t\boldsymbol{\xi}_t'\boldsymbol{\Lambda}\|^2] = O(Tn^{-2(2-\delta)})$;

(viii) $\mathsf{E}[\|(nT)^{-1}\sum_{t=1}^T \Delta\boldsymbol{\xi}_t\boldsymbol{\xi}_t'\|^2] = O(Tn^{-2(1-\delta)})$.

(ix) $\mathsf{E}[\|(n^{3/2}T^2)^{-1}\sum_{t=1}^T \boldsymbol{\xi}_t\boldsymbol{\xi}_t'\boldsymbol{\Lambda}\|^2] = O(n^{-(3-2\delta)})$.

Proof. Start with part (i):

$$\mathsf{E}\Big[\Big\|\frac{1}{nT^2}\sum_{t=1}^T \mathbf{F}_t\boldsymbol{\xi}_t'\boldsymbol{\Lambda}\Big\|^2\Big] = \frac{1}{n^2T^4}\sum_{j_1,j_2=1}^r \mathsf{E}\Big[\Big(\sum_{t=1}^T F_{j_1t}\sum_{i=1}^n \lambda_{ij_2}\xi_{it}\Big)^2\Big] \le \frac{C^2r}{n^2T^4}\sum_{t,s=1}^T\sum_{j=1}^r\sum_{i_1,i_2=1}^n \Big|\mathsf{E}\big[F_{jt}F_{js}\xi_{i_1t}\xi_{i_2s}\big]\Big|$$

$$\le \frac{C^2r}{n^2T^4}\sum_{t,s=1}^T\sum_{j=1}^r \Big|\mathsf{E}\big[F_{jt}F_{js}\big]\Big|\Big\{\sum_{i_1,i_2\in\mathcal{I}_1^c}\Big|\mathsf{E}\big[\xi_{i_1t}\xi_{i_2s}\big]\Big| + 3\sum_{i_1,i_2\in\mathcal{I}_1}\Big|\mathsf{E}\big[\xi_{i_1t}\xi_{i_2s}\big]\Big|\Big\}$$

$$\le \frac{C^2r}{n^2T^4}\sum_{t,s=1}^T\sum_{j=1}^r \mathsf{E}\big[F_{jt}^2\big]\Big\{\sum_{i_1,i_2\in\mathcal{I}_1^c}\Big|\mathsf{E}\big[\xi_{i_1t}\xi_{i_2s}\big]\Big| + 3\sum_{i_1,i_2\in\mathcal{I}_1}\Big|\mathsf{E}\big[\xi_{i_1t}\xi_{i_2s}\big]\Big|\Big\}$$

$$\le \frac{C^2r}{n^2T^4}\sum_{t,s=1}^T\sum_{j=1}^r \mathsf{E}\big[F_{jt}^2\big]K_2^2\Big\{\sum_{i_1,i_2\in\mathcal{I}_1^c}\Big|\mathsf{E}\big[\varepsilon_{i_1t}\varepsilon_{i_2t}\big]\Big| + 3\sum_{i_1,i_2\in\mathcal{I}_1}\sum_{s=1}^t\Big|\mathsf{E}\big[\varepsilon_{i_1s}\varepsilon_{i_2s}\big]\Big|\Big\}$$

$$\le \frac{C^2r^2}{n^2T^4}T\sum_{t=1}^T \mathsf{E}\big[F_{jt}^2\big]K_2^2M_3(n+n^\delta t) = O\left(\frac{1}{nT}\right) + O\left(\frac{1}{n^{2-\delta}}\right),$$

where we used Assumption 2(b) of uniformly bounded loadings, Assumption 3(a) and (e) of independent idiosyncratic shocks also independent of the common shocks, Assumptions 3 and 6 which bound the cross-sectional dependence of idiosyncratic components, square summability of the coefficients, with K_2 defined in (C7), Cauchy-Schwarz inequality, and Lemma D4(ii). This proves part (i).

For part (ii) we have:

$$\mathsf{E}\Big[\Big\|\frac{1}{\sqrt{n}T^2}\sum_{t=1}^T \mathbf{F}_t\boldsymbol{\xi}_t'\Big\|^2\Big] = \frac{1}{nT^4}\sum_{j=1}^r\sum_{i=1}^n \mathsf{E}\Big[\Big(\sum_{t=1}^T F_{jt}\xi_{it}\Big)^2\Big] \le \frac{1}{nT^4}\sum_{t,s=1}^T\sum_{j=1}^r\sum_{i_1,i_2=1}^n \Big|\mathsf{E}\big[F_{jt}F_{js}\xi_{i_1t}\xi_{i_2s}\big]\Big|$$

$$\le \frac{1}{nT^4}\sum_{t,s=1}^T\sum_{j=1}^r \mathsf{E}\big[F_{jt}^2\big]\Big\{\sum_{i_1,i_2\in\mathcal{I}_1^c}\Big|\mathsf{E}\big[\xi_{i_1t}\xi_{i_2s}\big]\Big| + 3\sum_{i_1,i_2\in\mathcal{I}_1}\Big|\mathsf{E}\big[\xi_{i_1t}\xi_{i_2s}\big]\Big|\Big\}$$

$$\le \frac{1}{nT^4}\sum_{t,s=1}^T\sum_{j=1}^r \mathsf{E}\big[F_{jt}^2\big]K_2^2\Big\{\sum_{i_1,i_2\in\mathcal{I}_1^c}\Big|\mathsf{E}\big[\varepsilon_{i_1t}\varepsilon_{i_2t}\big]\Big| + 3\sum_{i_1,i_2\in\mathcal{I}_1}\sum_{s=1}^t\Big|\mathsf{E}\big[\varepsilon_{i_1s}\varepsilon_{i_2s}\big]\Big|\Big\}$$

$$\le \frac{r}{nT^4}T\sum_{t=1}^T \mathsf{E}\big[F_{jt}^2\big]K_2^2M_3(n+n^\delta t) = O\left(\frac{1}{T}\right) + O\left(\frac{1}{n^{1-\delta}}\right),$$

using the same arguments used for proving part (i). This proves part (ii).

Turning to part (iii):

$$
\begin{aligned}
&\mathsf{E}\left[\left\|\frac{1}{n^2T^2}\sum_{t=1}^T \mathbf{\Lambda}'\boldsymbol{\xi}_t\boldsymbol{\xi}_t'\mathbf{\Lambda}\right\|^2\right] = \frac{1}{n^4T^4}\sum_{j_1,j_2=1}^r \mathsf{E}\left[\left(\sum_{t=1}^T\left(\sum_{i_1=1}^n \lambda_{i_1j_1}\xi_{i_1t}\right)\left(\sum_{i_2=1}^n \lambda_{i_2j_2}\xi_{i_2t}\right)\right)^2\right]\\
&\le \frac{C^4r^2}{n^4T^4}\sum_{t,s=1}^T\sum_{i_1,i_1'=1}^n\sum_{i_2,i_2'=1}^n \left|\mathsf{E}\left[\xi_{i_1t}\xi_{i_1't}\xi_{i_2s}\xi_{i_2's}\right]\right|\\
&\le \frac{C^4r^2K_2^4}{n^4T^4}\sum_{t,s=1}^T\left\{\sum_{i_1,i_1'\in\mathcal{I}_1^c}\sum_{i_2,i_2'\in\mathcal{I}_1^c}\left|\mathsf{E}\left[\varepsilon_{i_1t}\varepsilon_{i_1't}\varepsilon_{i_2s}\varepsilon_{i_2's}\right]\right| + 15\sum_{i_1,i_1'\in\mathcal{I}_1}\sum_{i_2,i_2'\in\mathcal{I}_1}\sum_{t_1',t_2'=1}^t\sum_{s_1',s_2'=1}^s\left|\mathsf{E}\left[\varepsilon_{i_1t_1'}\varepsilon_{i_1't_2'}\varepsilon_{i_2s_1'}\varepsilon_{i_2's_2'}\right]\right|\right\}\\
&\le \frac{C^4r^2K_2^4}{n^4T^4}\sum_{t,s=1}^T\left\{\sum_{i_1,i_1'\in\mathcal{I}_1}\left|\mathsf{E}\left[\varepsilon_{i_1t}\varepsilon_{i_1't}\right]\right|\sum_{i_2,i_2'\in\mathcal{I}_1}\left|\mathsf{E}\left[\varepsilon_{i_2s}\varepsilon_{i_2's}\right]\right| + 15\sum_{i_1,i_1'\in\mathcal{I}_1}\sum_{t'=1}^t\left|\mathsf{E}\left[\varepsilon_{i_1t'}\varepsilon_{i_1't'}\right]\right|\sum_{i_2,i_2'\in\mathcal{I}_1}\sum_{s'=1}^s\left|\mathsf{E}\left[\varepsilon_{i_2s'}\varepsilon_{i_2's'}\right]\right|\right\}\\
&\le \frac{C^4r^2K_2^4}{n^4T^4}\sum_{t,s=1}^T\left\{\left(\sum_{i_1,i_2\in\mathcal{I}_1^c}\left|\mathsf{E}\left[\varepsilon_{i_1t}\varepsilon_{i_2t}\right]\right|\right)^2 + 15\left(\sum_{i_1,i_2\in\mathcal{I}_1}\sum_{s=1}^t\left|\mathsf{E}\left[\varepsilon_{i_1s}\varepsilon_{i_2s}\right]\right|\right)^2\right\}\\
&\le \frac{C^4r^2K_2^4M_3^4}{n^4T^4}T^2(n^2+15n^{2\delta}t^2) = O\left(\frac{1}{n^2T^2}\right)+O\left(\frac{1}{n^{2(2-\delta)}}\right),
\end{aligned}
$$

using the same arguments used for proving part (i). This proves part (iii).

For part (iv) we have:

$$
\begin{aligned}
&\mathsf{E}\left[\left\|\frac{1}{nT^2}\sum_{t=1}^T\boldsymbol{\xi}_t\boldsymbol{\xi}_t'\right\|^2\right] = \frac{1}{n^2T^4}\sum_{i,j=1}^n\mathsf{E}\left[\left(\sum_{t=1}^T\xi_{it}\xi_{jt}\right)^2\right] \le \frac{1}{n^2T^4}\sum_{t,s=1}^T\left|\mathsf{E}\left[\xi_{it}\xi_{is}\xi_{jt}\xi_{js}\right]\right|\\
&\le \frac{K_2^4}{n^2T^4}\sum_{t,s=1}^T\left\{\left(\sum_{i_1,i_2\in\mathcal{I}_1^c}\left|\mathsf{E}\left[\varepsilon_{i_1t}\varepsilon_{i_2t}\right]\right|\right)^2 + 15\left(\sum_{i_1,i_2\in\mathcal{I}_1}\sum_{s=1}^t\left|\mathsf{E}\left[\varepsilon_{i_1s}\varepsilon_{i_2s}\right]\right|\right)^2\right\}\\
&\le \frac{K_2^4M_3^4}{n^2T^4}T^2(n^2+15n^{2\delta}t^2) = O\left(\frac{1}{T^2}\right)+O\left(\frac{1}{n^{2(1-\delta)}}\right),
\end{aligned}
$$

using the same arguments used for proving part (i). This proves part (iv). Parts (v) and (vi) follow from parts (i) and (ii) respectively. Parts (vii) and (ix) follow from part (iii), while part (viii) follows from part (iv). This completes the proof. □

Lemma D8 *Under Assumptions 1 through 5, as $n, T \to \infty$,*

(i) $\mathsf{E}[\|(nT^2)^{-1}\sum_{t=1}^T\mathbf{F}_t(\widehat{\mathbf{x}}_t-\mathbf{x}_t)'\mathbf{\Lambda}\|^2] = O(n^{-2(1-\eta)})$;
(ii) $\mathsf{E}[\|(\sqrt{n}T^2)^{-1}\sum_{t=1}^T\mathbf{F}_t(\widehat{\mathbf{x}}_t-\mathbf{x}_t)'\|^2] = O(n^{-(1-\eta)})$;
(iii) $\mathsf{E}[\|(n^2T^2)^{-1}\sum_{t=1}^T\mathbf{\Lambda}'(\widehat{\mathbf{x}}_t-\mathbf{x}_t)(\widehat{\mathbf{x}}_t-\mathbf{x}_t)'\mathbf{\Lambda}\|^2] = O(n^{-4(1-\eta)})$;
(iv) $\mathsf{E}[\|(nT^2)^{-1}\sum_{t=1}^T(\widehat{\mathbf{x}}_t-\mathbf{x}_t)(\widehat{\mathbf{x}}_t-\mathbf{x}_t)'\|^2] = O(n^{-2(1-\eta)})$;
(v) $\mathsf{E}[\|(nT)^{-1}\sum_{t=1}^T\Delta\mathbf{F}_t(\widehat{\mathbf{x}}_t-\mathbf{x}_t)'\mathbf{\Lambda}\|^2] = O(Tn^{-2(1-\eta)})$;
(vi) $\mathsf{E}[\|(\sqrt{n}T)^{-1}\sum_{t=1}^T\Delta\mathbf{F}_t(\widehat{\mathbf{x}}_t-\mathbf{x}_t)'\|^2] = O(Tn^{-(1-\eta)})$;
(vii) $\mathsf{E}[\|(n^2T)^{-1}\sum_{t=1}^T\mathbf{\Lambda}'(\Delta\widehat{\mathbf{x}}_t-\Delta\mathbf{x}_t)(\widehat{\mathbf{x}}_t-\mathbf{x}_t)'\mathbf{\Lambda}\|^2] = O(Tn^{-4(1-\eta)})$;
(viii) $\mathsf{E}[\|(nT)^{-1}\sum_{t=1}^T(\Delta\widehat{\mathbf{x}}_t-\Delta\mathbf{x}_t)(\widehat{\mathbf{x}}_t-\mathbf{x}_t)'\|^2] = O(Tn^{-2(1-\eta)})$.

Proof. We start with two preliminary results. First, note that for all $j = 1,\dots,r$ and all $t,s = 1,\dots,T$

we have

$$\mathsf{E}\big[F_{jt}^2F_{js}^2\big] \le q^4K_1^4\mathsf{E}\Bigg[\bigg(\sum_{t'=1}^{t}u_{jt'}\bigg)^2\bigg(\sum_{s'=1}^{s}u_{js'}\bigg)^2\Bigg] \le q^4K_1^4\sum_{t,t'=1}^{T}\sum_{s,s'=1}^{T}\Big|\mathsf{E}\big[u_{jt}u_{jt'}u_{js}u_{js'}\big]\Big|$$
$$\le q^4K_1^4\bigg\{\sum_{t=1}^{T}\mathsf{E}[u_{jt}^4]+\sum_{t,s=1}^{T}\mathsf{E}[u_{jt}^2u_{js}^2]\bigg\} \le q^4K_1^4M_1T^2, \tag{D32}$$

where we used square summability of the coefficients, with K_1 defined in (C5), and Assumption 1(a) of independence of the common shocks and finite fourth moments. Second, by using the same reasoning as in (C23) and (C24) in the proof of Lemma 1, we have that $\mathsf{E}[(\sum_{t=1}^{T}x_{it})^4]=O(T^6)$ and $\mathsf{E}[(\sum_{t=1}^{T}tx_{it})^4]=O(T^{10})$ for all $i=1,\dots,n$. Therefore,

$$\mathsf{E}[(\widehat{b}_i-b_i)^4]=\frac{\mathsf{E}\Big[\Big(\sum_{t=1}^{T}tx_{it}-\frac{T+1}{2}\sum_{t=1}^{T}x_{it}\Big)^4\Big]}{\big(\frac{1}{12}T(T^2-1)\big)^4}=\frac{C_1}{T^2}. \tag{D33}$$

for some positive real C_1 independent of i.

Now let us consider part (i):

$$\mathsf{E}\Bigg[\bigg\|\frac{1}{nT^2}\sum_{t=1}^{T}\mathbf{F}_t(\widehat{\mathbf{x}}_t-\mathbf{x}_t)'\boldsymbol{\Lambda}\bigg\|^2\Bigg]=\frac{1}{n^2T^4}\sum_{j_1,j_2=1}^{r}\mathsf{E}\Bigg[\bigg(\sum_{t=1}^{T}F_{j_1t}\sum_{i=1}^{n}\lambda_{ij_2}(b_i-\widehat{b}_i)t\bigg)^2\Bigg]$$
$$\le\frac{C^2r}{n^2T^4}\sum_{t,s=1}^{T}ts\sum_{j=1}^{r}\sum_{i_1,i_2\in\mathcal{I}_b}\Big|\mathsf{E}\big[F_{jt}F_{js}(b_{i_1}-\widehat{b}_{i_1})(b_{i_2}-\widehat{b}_{i_2})\big]\Big|$$
$$\le\frac{C^2r}{n^2T^4}\sum_{t,s=1}^{T}ts\sum_{j=1}^{r}\sqrt{\mathsf{E}\big[F_{jt}^2F_{js}^2\big]}\sum_{i_1,i_2\in\mathcal{I}_b}\sqrt{\mathsf{E}[(b_{i_1}-\widehat{b}_{i_1})^2(b_{i_2}-\widehat{b}_{i_2})^2]}$$
$$\le\frac{C^2r}{n^2T^4}\sum_{t,s=1}^{T}ts\sum_{j=1}^{r}\sqrt{\mathsf{E}\big[F_{jt}^2F_{js}^2\big]}\,n^\eta\sum_{i\in\mathcal{I}_b}\sqrt{\mathsf{E}[(b_i-\widehat{b}_i)^4]}$$
$$\le\frac{C^2r^2}{n^2T^4}\left(\frac{1}{12}T(T+1)(T+2)(3T+1)\right)q^2K_1^2\sqrt{M_1}Tn^{2\eta}\frac{\sqrt{C_1}}{T}=O\left(\frac{1}{n^{2(1-\eta)}}\right),$$

where we Assumption 2(b) of uniformly bounded loadings, Cauchy-Schwarz inequality, Assumption 5 (a) which bounds the number of deterministic linear trends, (D32), and (D33). This proves part (i).

For part (ii) we have

$$\mathsf{E}\Bigg[\bigg\|\frac{1}{\sqrt{n}T^2}\sum_{t=1}^{T}\mathbf{F}_t(\widehat{\mathbf{x}}_t-\mathbf{x}_t)'\bigg\|^2\Bigg]=\frac{1}{nT^4}\sum_{j=1}^{r}\sum_{i=1}^{n}\mathsf{E}\Bigg[\bigg(\sum_{t=1}^{T}F_{jt}(b_i-\widehat{b}_i)t\bigg)^2\Bigg]$$
$$\le\frac{1}{nT^4}\sum_{t,s=1}^{T}ts\sum_{j=1}^{r}\sum_{i\in\mathcal{I}_b}\Big|\mathsf{E}\big[F_{jt}F_{js}(b_i-\widehat{b}_i)^2\big]\Big|\le\frac{1}{nT^4}\sum_{t,s=1}^{T}ts\sum_{j=1}^{r}\sqrt{\mathsf{E}\big[F_{jt}^2F_{js}^2\big]}\sum_{i\in\mathcal{I}_b}\sqrt{\mathsf{E}[(b_i-\widehat{b}_i)^4]}$$
$$\le\frac{r}{nT^4}\left(\frac{1}{12}T(T+1)(T+2)(3T+1)\right)q^2K_1^2\sqrt{M_1}Tn^\eta\frac{\sqrt{C_1}}{T}=O\left(\frac{1}{n^{1-\eta}}\right),$$

using the same arguments used for proving part (i). This proves part (ii).

Turning to part (iii):

$$\mathsf{E}\left[\left\|\frac{1}{n^2T^2}\sum_{t=1}^T \mathbf{\Lambda}'(\widehat{\mathbf{x}}_t-\mathbf{x}_t)(\widehat{\mathbf{x}}_t-\mathbf{x}_t)'\mathbf{\Lambda}\right\|^2\right]=\frac{1}{n^4T^4}\sum_{j_1,j_2=1}^r \mathsf{E}\left[\left(\sum_{t=1}^T\left(\sum_{i_1=1}^n \lambda_{i_1j_1}(b_{i_1}-\widehat{b}_{i_1})t\right)\left(\sum_{i_2=1}^n \lambda_{i_2j_2}(b_{i_2}-\widehat{b}_{i_2})t\right)\right)^2\right]$$

$$\le \frac{C^4r^2}{n^4T^4}\sum_{t,s=1}^T t^2s^2n^{2\eta}\sum_{i,j\in\mathcal{I}_b}\mathsf{E}[(b_i-\widehat{b}_i)^2(b_j-\widehat{b}_j)^2]\le \frac{C^4r^2}{n^4T^4}\sum_{t,s=1}^T t^2s^2n^{3\eta}\sum_{i\in\mathcal{I}_b}\mathsf{E}[(b_i-\widehat{b}_i)^4]$$

$$\le \frac{C^4r^2}{n^4T^4}\left(\frac{1}{30}T(T+1)(T+2)(2T+1)(2T+3)(5T-1)\right)n^{4\eta}\frac{C_1}{T^2}=O\left(\frac{1}{n^{4(1-\eta)}}\right),$$

using the same arguments used for proving part (i). This proves part (iii).

For part (iv) we have:

$$\mathsf{E}\left[\left\|\frac{1}{nT^2}\sum_{t=1}^T(\widehat{\mathbf{x}}_t-\mathbf{x}_t)(\widehat{\mathbf{x}}_t-\mathbf{x}_t)'\right\|^2\right]=\frac{1}{n^2T^4}\sum_{i,j=1}^n \mathsf{E}\left[\left(\sum_{t=1}^T(b_i-\widehat{b}_i)(b_j-\widehat{b}_j)t^2\right)^2\right]$$

$$\le \frac{1}{n^2T^4}\sum_{t,s=1}^T t^2s^2\sum_{i,j=1}^n \mathsf{E}[(b_i-\widehat{b}_i)^2(b_j-\widehat{b}_j)^2]\le \frac{1}{n^2T^4}\sum_{t,s=1}^T t^2s^2n^\eta\sum_{i=1}^n\mathsf{E}[(b_i-\widehat{b}_i)^4]$$

$$\le \frac{1}{n^2T^4}\left(\frac{1}{30}T(T+1)(T+2)(2T+1)(2T+3)(5T-1)\right)n^{2\eta}\frac{C_1}{T^2}=O\left(\frac{1}{n^{2(1-\eta)}}\right),$$

using the same arguments used for proving part (i). This proves part (iv). Parts (v) and (vi) follow from parts (i) and (ii) respectively. Part (vii) follows from part (iii), while part (viii) follows from part (iv). This completes the proof. □

Lemma D9 *Under Assumptions 1 through 6, as $n,T\to\infty$,*
(i) $\mathsf{E}[\|(n^2T^2)^{-1}\sum_{t=1}^T \mathbf{\Lambda}'\boldsymbol{\xi}_t(\widehat{\mathbf{x}}_t-\mathbf{x}_t)'\mathbf{\Lambda}\|^2]=O(n^{-2(2-\delta-\eta)})$;
(ii) $\mathsf{E}[\|(nT^2)^{-1}\sum_{t=1}^T \boldsymbol{\xi}_t(\widehat{\mathbf{x}}_t-\mathbf{x}_t)'\|^2]=O(n^{-(2-\delta-\eta)})$.

Proof. First, note that for all $i,j\in\mathcal{I}_i$ and all $t,s=1,\dots,T$ we have

$$\mathsf{E}\left[\xi_{it}^2\xi_{js}^2\right]\le K_2^4\mathsf{E}\left[\left(\sum_{t'=1}^t\varepsilon_{it'}\right)^2\left(\sum_{s'=1}^s\varepsilon_{js'}\right)^2\right]\le K_2^4\sum_{t,t'=1}^T\sum_{s,s'=1}^T\left|\mathsf{E}\left[\varepsilon_{it}\varepsilon_{it'}\varepsilon_{js}\varepsilon_{js'}\right]\right|$$

$$\le K_2^4\left\{\sum_{t=1}^T\mathsf{E}[\varepsilon_{it}^2\varepsilon_{jt}^2]+\sum_{t,s=1}^T\mathsf{E}[\varepsilon_{it}^2\varepsilon_{js}^2]\right\}\le K_2^4M_2T^2, \tag{D34}$$

where we used square summability of the coefficients, with K_2 defined in (C7), and Assumption 3(a) of independence of the idiosyncratic shocks and finite fourth moments.

Then, consider part (i):

$$\mathsf{E}\left[\left\|\frac{1}{n^2T^2}\sum_{t=1}^T \mathbf{\Lambda}'\boldsymbol{\xi}_t(\widehat{\mathbf{x}}_t-\mathbf{x}_t)'\mathbf{\Lambda}\right\|^2\right]=\frac{1}{n^4T^4}\sum_{j_1,j_2=1}^r\mathsf{E}\left[\left(\sum_{t=1}^T\left(\sum_{i_1=1}^n\lambda_{i_1j_1}\xi_{i_1t}\right)\left(\sum_{i_2=1}^n\lambda_{i_2j_2}(b_{i_2}-\widehat{b}_{i_2})t\right)\right)^2\right]$$

$$\le \frac{C^4r^2}{n^4T^4}\sum_{t,s=1}^T ts\sum_{i_1,i_1'=1}^n\sum_{i_2,i_2'\in\mathcal{I}_b}\left|\mathsf{E}\left[\xi_{i_1t}\xi_{i_1's}(b_{i_2}-\widehat{b}_{i_2})(b_{i_2'}-\widehat{b}_{i_2'})\right]\right|$$

$$\le \frac{C^4r^2}{n^4T^4}\sum_{t,s=1}^T ts\left\{\sum_{i_1,i_1'\in\mathcal{I}_1^c}\sqrt{\mathsf{E}\left[\xi_{i_1t}^2\xi_{i_1's}^2\right]}+3\sum_{i_1,i_1'\in\mathcal{I}_1}\sqrt{\mathsf{E}\left[\xi_{i_1t}^2\xi_{i_1's}^2\right]}\right\}n^\eta\sum_{i_2\in\mathcal{I}_b}\sqrt{\mathsf{E}[(b_{i_2}-\widehat{b}_{i_2})^4]}$$

$$\le \frac{C^4r^2}{n^4T^4}\left(\frac{1}{12}T(T+1)(T+2)(3T+1)\right)K_2^2\sqrt{M_2}(n^2+n^{2\delta}T)n^{2\eta}\frac{\sqrt{C_1}}{T}=O\left(\frac{1}{n^{2(1-\eta)}T}\right)+O\left(\frac{1}{n^{2(2-\delta-\eta)}}\right),$$

where we Assumption 2(b) of uniformly bounded loadings, Cauchy-Schwarz inequality, Assumption 5 (a) which bounds the number of deterministic linear trends, Assumption 6 which bounds the number of $I(1)$ idiosyncratic components, (D34), and (D33) in the proof of Lemma D8. This proves part (i).

For part (ii) we have:

$$\mathsf{E}\left[\left\|\frac{1}{nT^2}\sum_{t=1}^T \boldsymbol{\xi}_t(\widehat{\mathbf{x}}_t-\mathbf{x}_t)'\right\|^2\right] = \frac{1}{n^2T^4}\sum_{i,j=1}^n \mathsf{E}\left[\left(\sum_{t=1}^T \xi_{it}(b_j-\widehat{b}_j)t\right)^2\right]$$

$$\leq \frac{1}{n^2T^4}\sum_{t,s=1}^T ts\sum_{i,j=1}^n \left|\mathsf{E}\left[\xi_{it}\xi_{is}(b_j-\widehat{b}_j)^2\right]\right|$$

$$\leq \frac{1}{n^2T^4}\sum_{t,s=1}^T ts\left\{\sum_{i\in\mathcal{I}_1^c}\sqrt{\mathsf{E}\left[\xi_{it}^2\xi_{is}^2\right]}+3\sum_{i\in\mathcal{I}_1}\sqrt{\mathsf{E}\left[\xi_{it}^2\xi_{is}^2\right]}\right\}\sum_{j\in\mathcal{I}_b}\sqrt{\mathsf{E}[(b_j-\widehat{b}_j)^4]}$$

$$\leq \frac{1}{n^2T^4}\left(\frac{1}{12}T(T+1)(T+2)(3T+1)\right)K_2^2\sqrt{M_2}(n+n^\delta T)n^\eta\frac{\sqrt{C_1}}{T} = O\left(\frac{1}{n^{(1-\eta)}T}\right)+O\left(\frac{1}{n^{2-\delta-\eta}}\right),$$

using the same arguments used for proving part (i). This proves part (ii). □

Lemma D10 *Define the matrices*

$$\widehat{\mathbf{M}}_{00}=\frac{1}{T}\sum_{t=1}^T\Delta\widehat{\mathbf{F}}_t\Delta\widehat{\mathbf{F}}_t',\quad \widehat{\mathbf{M}}_{01}=\frac{1}{T}\sum_{t=1}^T\Delta\widehat{\mathbf{F}}_t\widehat{\mathbf{F}}_{t-1}',\quad \widehat{\mathbf{M}}_{02}=\frac{1}{T}\sum_{t=1}^T\Delta\widehat{\mathbf{F}}_t\Delta\widehat{\mathbf{F}}_{t-1}',$$
$$\widehat{\mathbf{M}}_{11}=\frac{1}{T}\sum_{t=1}^T\widehat{\mathbf{F}}_t\widehat{\mathbf{F}}_t',\quad \widehat{\mathbf{M}}_{21}=\frac{1}{T}\sum_{t=1}^T\Delta\widehat{\mathbf{F}}_{t-1}'\widehat{\mathbf{F}}_{t-1},\quad \widehat{\mathbf{M}}_{22}=\frac{1}{T}\sum_{t=1}^T\Delta\widehat{\mathbf{F}}_{t-1}\Delta\widehat{\mathbf{F}}_{t-1}',$$

and denote by $\mathbf{M}_{ij}$, for $i,j=0,1,2$, the analogous ones but computed by using $\check{\mathbf{F}}_t=\mathbf{J}\mathbf{F}_t$. Define also $\check{\boldsymbol{\beta}}=\mathbf{J}\boldsymbol{\beta}$. Under Assumptions 1 through 5, as $n,T\to\infty$,

(i) $\|T^{-1}\widehat{\mathbf{M}}_{11}-T^{-1}\mathbf{M}_{11}\|=O_p(\max(n^{-1/2},T^{-1/2},n^{-(1-\eta)}))$;
(ii) $\|\widehat{\mathbf{M}}_{00}-\mathbf{M}_{00}\|=O_p(\max(n^{-1/2},T^{-1/2},n^{-(1-\eta)}))$;
(iii) $\|\widehat{\mathbf{M}}_{02}-\mathbf{M}_{02}\|=O_p(\max(n^{-1/2},T^{-1/2},n^{-(1-\eta)}))$;
(iv) $\|\widehat{\mathbf{M}}_{22}-\mathbf{M}_{22}\|=O_p(\max(n^{-1/2},T^{-1/2},n^{-(1-\eta)}))$.
If also Assumption 6 holds, then,
(v) $\|\widehat{\mathbf{M}}_{01}\check{\boldsymbol{\beta}}-\mathbf{M}_{01}\check{\boldsymbol{\beta}}\|=O_p(\vartheta_{nT,\delta,\eta})$;
(vi) $\|\widehat{\mathbf{M}}_{21}\check{\boldsymbol{\beta}}-\mathbf{M}_{21}\check{\boldsymbol{\beta}}\|=O_p(\vartheta_{nT,\delta,\eta})$;
(vii) $\|T^{-1/2}\widehat{\mathbf{M}}_{01}-T^{-1/2}\mathbf{M}_{01}\|=O_p(\vartheta_{nT,\delta,\eta})$;
(viii) $\|T^{-1/2}\widehat{\mathbf{M}}_{21}-T^{-1/2}\mathbf{M}_{21}\|=O_p(\vartheta_{nT,\delta,\eta})$;
(ix) $\|\check{\boldsymbol{\beta}}'\widehat{\mathbf{M}}_{11}\check{\boldsymbol{\beta}}-\check{\boldsymbol{\beta}}'\mathbf{M}_{11}\check{\boldsymbol{\beta}}\|=O_p(\vartheta_{nT,\delta,\eta})$.

Proof. Throughout, we use $\|\boldsymbol{\beta}\|=O(1)$ and obviously $\|\mathbf{J}\|=1$ and the fact that, since $\sqrt{T}/n\to 0$, as $n,T\to\infty$ we have (see also (C25) and (C26) in the proof of Lemma 1)

$$\left\|\frac{\widehat{\boldsymbol{\Lambda}}-\boldsymbol{\Lambda}\mathbf{J}}{\sqrt{n}}\right\|=O_p\left(\frac{1}{\sqrt{T}}\right)\ \text{ and }\ \left\|\frac{\widehat{\boldsymbol{\Lambda}}'\boldsymbol{\Lambda}}{n}-\mathbf{J}\right\|=O_p\left(\frac{1}{\sqrt{T}}\right). \tag{D35}$$

and therefore $\|n^{-1}\widehat{\boldsymbol{\Lambda}}'\boldsymbol{\Lambda}\|=O_p(1)$.

Start with part (i). By adding and subtracting $\mathbf{J}\mathbf{F}_t$ from $\widehat{\mathbf{F}}_t$, we have

$$\left\|\frac{1}{T^2}\sum_{t=1}^T \widehat{\mathbf{F}}_t\widehat{\mathbf{F}}_t' - \frac{1}{T^2}\sum_{t=1}^T \check{\mathbf{F}}_t\check{\mathbf{F}}_t'\right\| \leq 2\left\|\frac{1}{T^2}\sum_{t=1}^T\left(\widehat{\mathbf{F}}_t - \mathbf{J}\mathbf{F}_t\right)\left(\mathbf{J}\mathbf{F}_t\right)'\right\| + \left\|\frac{1}{T^2}\sum_{t=1}^T\left(\widehat{\mathbf{F}}_t - \mathbf{J}\mathbf{F}_t\right)\left(\widehat{\mathbf{F}}_t - \mathbf{J}\mathbf{F}_t\right)'\right\|. \tag{D36}$$

Using (7) and (18), the first term on the rhs of (D36) is such that

$$\begin{aligned}
\left\|\frac{1}{T^2}\sum_{t=1}^T\left(\widehat{\mathbf{F}}_t - \mathbf{J}\mathbf{F}_t\right)\left(\mathbf{J}\mathbf{F}_t\right)'\right\| &= \left\|\frac{1}{T^2}\sum_{t=1}^T\left(\frac{\widehat{\mathbf{\Lambda}}'\widehat{\mathbf{x}}_t}{n} - \mathbf{J}\mathbf{F}_t\right)\left(\mathbf{J}\mathbf{F}_t\right)'\right\| \\
&= \left\|\frac{1}{T^2}\sum_{t=1}^T\left(\frac{\widehat{\mathbf{\Lambda}}'\mathbf{\Lambda}\mathbf{F}_t}{n} - \mathbf{J}\mathbf{F}_t + \frac{\widehat{\mathbf{\Lambda}}'\boldsymbol{\xi}_t}{n} + \frac{\widehat{\mathbf{\Lambda}}'(\widehat{\mathbf{x}}_t - \mathbf{x}_t)}{n}\right)\left(\mathbf{J}\mathbf{F}_t\right)'\right\| \qquad \text{(D37)} \\
&\leq \underbrace{\left\|\frac{1}{T^2}\sum_{t=1}^T\left(\frac{\widehat{\mathbf{\Lambda}}'\mathbf{\Lambda}\mathbf{F}_t}{n} - \mathbf{J}\mathbf{F}_t\right)\left(\mathbf{J}\mathbf{F}_t\right)'\right\|}_{\mathcal{A}_1} + \underbrace{\left\|\frac{1}{T^2}\sum_{t=1}^T\frac{\widehat{\mathbf{\Lambda}}'\boldsymbol{\xi}_t\mathbf{F}_t'\mathbf{J}}{n}\right\|}_{\mathcal{B}_1} + \underbrace{\left\|\frac{1}{T^2}\sum_{t=1}^T\frac{\widehat{\mathbf{\Lambda}}'(\widehat{\mathbf{x}}_t - \mathbf{x}_t)\mathbf{F}_t'\mathbf{J}}{n}\right\|}_{\mathcal{C}_1}
\end{aligned}$$

Now, consider each of the three terms in (D37) separately:

$$\mathcal{A}_1 \leq \left\|\frac{\widehat{\mathbf{\Lambda}}'\mathbf{\Lambda}}{n} - \mathbf{J}\right\| \left\|\frac{1}{T^2}\sum_{t=1}^T \mathbf{F}_t\mathbf{F}_t'\right\| = O_p\left(\frac{1}{\sqrt{T}}\right),$$

because of (D35) and Lemma D6(ii). Then, considering the worst case, i.e. $\delta = 1$, we have

$$\mathcal{B}_1 \leq \left\|\frac{\widehat{\mathbf{\Lambda}} - \mathbf{\Lambda}\mathbf{J}}{\sqrt{n}}\right\| \left\|\frac{1}{T^2}\sum_{t=1}^T \frac{\boldsymbol{\xi}_t\mathbf{F}_t'}{\sqrt{n}}\right\| + \left\|\frac{1}{T^2}\sum_{t=1}^T \frac{\mathbf{\Lambda}'\boldsymbol{\xi}_t\mathbf{F}_t'}{n}\right\| = O_p\left(\max\left(\frac{1}{\sqrt{T}}, \frac{1}{\sqrt{n}}\right)\right),$$

because of (D35) and Lemma D7(i) and D7(ii). Last,

$$\mathcal{C}_1 \leq \left\|\frac{\widehat{\mathbf{\Lambda}} - \mathbf{\Lambda}\mathbf{J}}{\sqrt{n}}\right\| \left\|\frac{1}{T^2}\sum_{t=1}^T \frac{(\widehat{\mathbf{x}}_t - \mathbf{x}_t)\mathbf{F}_t'}{\sqrt{n}}\right\| + \left\|\frac{1}{T^2}\sum_{t=1}^T \frac{\mathbf{\Lambda}'(\widehat{\mathbf{x}}_t - \mathbf{x}_t)\mathbf{F}_t'}{n}\right\| = O_p\left(\frac{1}{n^{(1-\eta)/2}\sqrt{T}}\right) + O_p\left(\frac{1}{n^{1-\eta}}\right),$$

because of (D35) and Lemma D8(i) and D8(ii).

Consider the second term on the rhs of (D37)

$$\begin{aligned}
&\left\|\frac{1}{T^2}\sum_{t=1}^T\left(\widehat{\mathbf{F}}_t - \mathbf{J}\mathbf{F}_t\right)\left(\widehat{\mathbf{F}}_t - \mathbf{J}\mathbf{F}_t\right)'\right\| = \left\|\frac{1}{T^2}\sum_{t=1}^T\left(\frac{\widehat{\mathbf{\Lambda}}'\widehat{\mathbf{x}}_t}{n} - \mathbf{J}\mathbf{F}_t\right)\left(\frac{\widehat{\mathbf{\Lambda}}'\widehat{\mathbf{x}}_t}{n} - \mathbf{J}\mathbf{F}_t\right)'\right\| \\
&= \left\|\frac{1}{T^2}\sum_{t=1}^T\left(\frac{\widehat{\mathbf{\Lambda}}'\mathbf{\Lambda}\mathbf{F}_t}{n} - \mathbf{J}\mathbf{F}_t + \frac{\widehat{\mathbf{\Lambda}}'\boldsymbol{\xi}_t}{n} + \frac{\widehat{\mathbf{\Lambda}}'(\widehat{\mathbf{x}}_t - \mathbf{x}_t)}{n}\right)\left(\frac{\widehat{\mathbf{\Lambda}}'\mathbf{\Lambda}\mathbf{F}_t}{n} - \mathbf{J}\mathbf{F}_t + \frac{\widehat{\mathbf{\Lambda}}'\boldsymbol{\xi}_t}{n} + \frac{\widehat{\mathbf{\Lambda}}'(\widehat{\mathbf{x}}_t - \mathbf{x}_t)}{n}\right)'\right\| \\
&\leq \underbrace{\left\|\frac{1}{T^2}\sum_{t=1}^T\frac{\widehat{\mathbf{\Lambda}}'\mathbf{\Lambda}\mathbf{F}_t\mathbf{F}_t'}{n}\left(\frac{\mathbf{\Lambda}'\widehat{\mathbf{\Lambda}}}{n} - \mathbf{J}\right) + \mathbf{J}\mathbf{F}_t\mathbf{F}_t'\left(\mathbf{J} - \frac{\mathbf{\Lambda}'\widehat{\mathbf{\Lambda}}}{n}\right)\right\|}_{\mathcal{D}_1} + 2\underbrace{\left\|\frac{1}{T^2}\sum_{t=1}^T\frac{\widehat{\mathbf{\Lambda}}'\mathbf{\Lambda}\mathbf{F}_t\boldsymbol{\xi}_t'\widehat{\mathbf{\Lambda}}}{n^2}\right\|}_{\mathcal{E}_1} \\
&\quad + 2\underbrace{\left\|\frac{1}{T^2}\sum_{t=1}^T\frac{\widehat{\mathbf{\Lambda}}'\boldsymbol{\xi}_t\mathbf{F}_t'\mathbf{J}}{n}\right\|}_{\mathcal{F}_1} + \underbrace{\left\|\frac{1}{T^2}\sum_{t=1}^T\frac{\widehat{\mathbf{\Lambda}}'\boldsymbol{\xi}_t\boldsymbol{\xi}_t'\widehat{\mathbf{\Lambda}}}{n^2}\right\|}_{\mathcal{G}_1} + 2\underbrace{\left\|\frac{1}{T^2}\sum_{t=1}^T\left(\frac{\widehat{\mathbf{\Lambda}}'\mathbf{\Lambda}}{n} - \mathbf{J}\right)\frac{\mathbf{F}_t(\widehat{\mathbf{x}}_t - \mathbf{x}_t)'\widehat{\mathbf{\Lambda}}}{n}\right\|}_{\mathcal{H}_1} \\
&\quad + 2\underbrace{\left\|\frac{1}{T^2}\sum_{t=1}^T\frac{\widehat{\mathbf{\Lambda}}'\boldsymbol{\xi}_t(\widehat{\mathbf{x}}_t - \mathbf{x}_t)'\widehat{\mathbf{\Lambda}}}{n^2}\right\|}_{\mathcal{J}_1} + \underbrace{\left\|\frac{1}{T^2}\sum_{t=1}^T\frac{\widehat{\mathbf{\Lambda}}'(\widehat{\mathbf{x}}_t - \mathbf{x}_t)(\widehat{\mathbf{x}}_t - \mathbf{x}_t)'\widehat{\mathbf{\Lambda}}}{n^2}\right\|}_{\mathcal{K}_1}. \qquad \text{(D38)}
\end{aligned}$$

Now, consider each of the terms in (D38) separately. Term $\mathcal{D}_1$ behaves like $\mathcal{A}_1$, $\mathcal{E}_1$ and $\mathcal{F}_1$ behave like $\mathcal{B}_1$. Then term $\mathcal{H}_1$ is dominated by $\mathcal{C}_1$. Moreover, by Lemma D9(i) and D9(ii) term $\mathcal{J}_1$ is dominated by $\mathcal{H}_1$ and by Lemma D8(iii) and D8(iv) term $\mathcal{K}_1$ is also dominated by $\mathcal{H}_1$. We are left with $\mathcal{G}_1$, which, considering the worst case, i.e. $\delta = 1$, is such that

$$\mathcal{G}_1 \leq \left\| \frac{\widehat{\mathbf{\Lambda}} - \mathbf{\Lambda J}}{\sqrt{n}} \right\|^2 \left\| \frac{1}{T^2} \sum_{t=1}^T \frac{\boldsymbol{\xi}_t \boldsymbol{\xi}_t'}{n} \right\| + \left\| \frac{1}{T^2} \sum_{t=1}^T \frac{\mathbf{\Lambda}' \boldsymbol{\xi}_t \boldsymbol{\xi}_t' \mathbf{\Lambda}}{n^2} \right\|$$
$$+ 2 \left\| \frac{\widehat{\mathbf{\Lambda}} - \mathbf{\Lambda J}}{\sqrt{n}} \right\| \left\| \frac{1}{T^2} \sum_{t=1}^T \frac{\boldsymbol{\xi}_t \boldsymbol{\xi}_t'}{n} \right\| \left\| \frac{\mathbf{\Lambda}}{\sqrt{n}} \right\| = O_p \left(\max \left(\frac{1}{\sqrt{T}}, \frac{1}{n} \right) \right),$$

because of (D35) and Lemma D7(iii) and D7(iv). By substituting (D37) and (D38) into (D36), we prove part (i). Part (ii), (iii), (iv) are proved analogously by noting that since in these cases we deal with differenced data the terms due to the de-trending are all $O_p(T^{-1/2})$ (this can be proved by simple modifications in the proof of Lemma D8).

Now, consider part (v):

$$\left\| \frac{1}{T} \sum_{t=1}^T \Delta \widehat{\mathbf{F}}_t \widehat{\mathbf{F}}_{t-1}' \check{\boldsymbol{\beta}} - \frac{1}{T} \sum_{t=1}^T \Delta \check{\mathbf{F}}_t \check{\mathbf{F}}_{t-1}' \check{\boldsymbol{\beta}} \right\| \leq \left\| \frac{1}{T} \sum_{t=1}^T \Big(\mathbf{J} \Delta \mathbf{F}_t \Big) \Big(\widehat{\mathbf{F}}_{t-1} - \mathbf{J F}_{t-1} \Big)' \check{\boldsymbol{\beta}} \right\|$$
$$+ \left\| \frac{1}{T} \sum_{t=1}^T \Big(\Delta \widehat{\mathbf{F}}_t - \mathbf{J} \Delta \mathbf{F}_t \Big) \Big(\check{\boldsymbol{\beta}}' \mathbf{J F}_{t-1} \Big)' \right\| + \left\| \frac{1}{T} \sum_{t=1}^T \Big(\Delta \widehat{\mathbf{F}}_t - \mathbf{J} \Delta \mathbf{F}_t \Big) \Big(\widehat{\mathbf{F}}_{t-1} - \mathbf{J F}_{t-1} \Big)' \check{\boldsymbol{\beta}} \right\|. \quad \text{(D39)}$$

Consider the first term on the rhs of (D39)

$$\left\| \frac{1}{T} \sum_{t=1}^T \Big(\mathbf{J} \Delta \mathbf{F}_t \Big) \Big(\widehat{\mathbf{F}}_{t-1} - \mathbf{J F}_{t-1} \Big)' \check{\boldsymbol{\beta}} \right\| = \left\| \frac{1}{T} \sum_{t=1}^T \Big(\mathbf{J} \Delta \mathbf{F}_t \Big) \Big(\frac{\widehat{\mathbf{\Lambda}}' \widehat{\mathbf{x}}_{t-1}}{n} - \mathbf{J F}_{t-1} \Big)' \check{\boldsymbol{\beta}} \right\|$$
$$= \left\| \frac{1}{T} \sum_{t=1}^T \Big(\mathbf{J} \Delta \mathbf{F}_t \Big) \Big(\frac{\widehat{\mathbf{\Lambda}}' \mathbf{\Lambda F}_{t-1}}{n} - \mathbf{J F}_{t-1} + \frac{\widehat{\mathbf{\Lambda}}' \boldsymbol{\xi}_{t-1}}{n} + \frac{\widehat{\mathbf{\Lambda}}' (\widehat{\mathbf{x}}_{t-1} - \mathbf{x}_{t-1})}{n} \Big)' \check{\boldsymbol{\beta}} \right\| \quad \text{(D40)}$$
$$= \underbrace{\left\| \frac{1}{T} \sum_{t=1}^T \Big(\mathbf{J} \Delta \mathbf{F}_t \Big) \Big(\frac{\widehat{\mathbf{\Lambda}}' \mathbf{\Lambda F}_{t-1}}{n} - \mathbf{J F}_{t-1} \Big)' \check{\boldsymbol{\beta}} \right\|}_{\mathcal{A}_2} + \underbrace{\left\| \frac{1}{T} \sum_{t=1}^T \frac{\mathbf{J} \Delta \mathbf{F}_t \boldsymbol{\xi}_{t-1}' \widehat{\mathbf{\Lambda}} \check{\boldsymbol{\beta}}}{n} \right\|}_{\mathcal{B}_2}$$
$$+ \underbrace{\left\| \frac{1}{T} \sum_{t=1}^T \frac{\mathbf{J} \Delta \mathbf{F}_t (\widehat{\mathbf{x}}_{t-1} - \mathbf{x}_{t-1})' \widehat{\mathbf{\Lambda}} \check{\boldsymbol{\beta}}}{n} \right\|}_{\mathcal{C}_2}.$$

Now, consider each of the three terms in (D40) separately:

$$\mathcal{A}_2 \leq \left\| \frac{\widehat{\mathbf{\Lambda}}' \mathbf{\Lambda}}{n} - \mathbf{J} \right\| \left\| \frac{1}{T} \sum_{t=1}^T \Delta \mathbf{F}_t \mathbf{F}_{t-1}' \right\| \|\check{\boldsymbol{\beta}}\| = O_p \left(\frac{1}{\sqrt{T}} \right),$$

because of (D35) and Lemma D6(iii). Then,

$$\mathcal{B}_2 \leq \left\| \frac{\widehat{\mathbf{\Lambda}} - \mathbf{\Lambda J}}{\sqrt{n}} \right\| \left\| \frac{1}{T} \sum_{t=1}^T \frac{\Delta \mathbf{F}_t \boldsymbol{\xi}_{t-1}'}{\sqrt{n}} \right\| + \left\| \frac{1}{T} \sum_{t=1}^T \frac{\Delta \mathbf{F}_t \mathbf{\Lambda}' \boldsymbol{\xi}_{t-1}}{n} \right\| \|\check{\boldsymbol{\beta}}\| = O_p \left(\max \left(\frac{1}{n^{(1-\delta)/2}}, \frac{\sqrt{T}}{n^{(2-\delta)/2}} \right) \right),$$

because of (D35) and Lemma D7(v) and D7(vi). Last,

$$\mathcal{C}_2 \leq \left\| \frac{\widehat{\mathbf{\Lambda}} - \mathbf{\Lambda J}}{\sqrt{n}} \right\| \left\| \frac{1}{T} \sum_{t=1}^T \frac{\Delta \mathbf{F}_t (\widehat{\mathbf{x}}_{t-1} - \mathbf{x}_{t-1})'}{\sqrt{n}} \right\| + \left\| \frac{1}{T} \sum_{t=1}^T \frac{\Delta \mathbf{F}_t (\widehat{\mathbf{x}}_{t-1} - \mathbf{x}_{t-1})' \mathbf{\Lambda}}{n} \right\| = O_p \left(\frac{1}{n^{(1-\eta)/2}}, \frac{\sqrt{T}}{n^{1-\eta}} \right),$$

because of (D35) and Lemma D8(v) and D8(vi). The second term on the rhs of (D39) contains only stationary terms, thus is dominated by the first one.

Then, consider the third term on the rhs of (D39)

$$\left\|\frac{1}{T}\sum_{t=1}^{T}\Big(\Delta\widehat{\mathbf{F}}_t-\mathbf{J}\Delta\mathbf{F}_t\Big)\Big(\check{\boldsymbol\beta}'\widehat{\mathbf{F}}_{t-1}-\check{\boldsymbol\beta}'\mathbf{J}\mathbf{F}_{t-1}\Big)'\right\|=\left\|\frac{1}{T}\sum_{t=1}^{T}\left(\frac{\widehat{\boldsymbol\Lambda}'\Delta\widehat{\mathbf{x}}_t}{n}-\mathbf{J}\Delta\mathbf{F}_t\right)\left(\frac{\widehat{\boldsymbol\Lambda}'\widehat{\mathbf{x}}_{t-1}}{n}-\mathbf{J}\mathbf{F}_{t-1}\right)'\check{\boldsymbol\beta}\right\|$$

$$\leq\underbrace{\left\|\frac{1}{T}\sum_{t=1}^{T}\frac{\widehat{\boldsymbol\Lambda}'\boldsymbol\Lambda\Delta\mathbf{F}_t\mathbf{F}_{t-1}'}{n}\left(\frac{\boldsymbol\Lambda'\widehat{\boldsymbol\Lambda}}{n}-\mathbf{J}\right)\check{\boldsymbol\beta}+\mathbf{J}\Delta\mathbf{F}_t\mathbf{F}_{t-1}'\left(\mathbf{J}-\frac{\boldsymbol\Lambda'\widehat{\boldsymbol\Lambda}}{n}\right)\check{\boldsymbol\beta}\right\|}_{\mathcal{D}_2}+\underbrace{\left\|\frac{1}{T}\sum_{t=1}^{T}\frac{\widehat{\boldsymbol\Lambda}'\boldsymbol\Lambda\Delta\mathbf{F}_t\boldsymbol\xi_{t-1}'\widehat{\boldsymbol\Lambda}\check{\boldsymbol\beta}}{n^2}\right\|}_{\mathcal{E}_2}$$

$$+\underbrace{\left\|\frac{1}{T}\sum_{t=1}^{T}\frac{\widehat{\boldsymbol\Lambda}'\Delta\boldsymbol\xi_t\mathbf{F}_{t-1}'\boldsymbol\Lambda'\widehat{\boldsymbol\Lambda}\check{\boldsymbol\beta}}{n^2}\right\|}_{\mathcal{F}_2}+\underbrace{\left\|\frac{1}{T}\sum_{t=1}^{T}\frac{\mathbf{J}\Delta\mathbf{F}_t\boldsymbol\xi_{t-1}'\widehat{\boldsymbol\Lambda}\check{\boldsymbol\beta}}{n}\right\|}_{\mathcal{G}_2}+\underbrace{\left\|\frac{1}{T}\sum_{t=1}^{T}\frac{\widehat{\boldsymbol\Lambda}'\Delta\boldsymbol\xi_t\mathbf{F}_{t-1}'\mathbf{J}\check{\boldsymbol\beta}}{n}\right\|}_{\mathcal{H}_2}$$

$$+\underbrace{\left\|\frac{1}{T}\sum_{t=1}^{T}\frac{\widehat{\boldsymbol\Lambda}'\Delta\boldsymbol\xi_t\boldsymbol\xi_{t-1}'\widehat{\boldsymbol\Lambda}\check{\boldsymbol\beta}}{n^2}\right\|}_{\mathcal{J}_2}+\underbrace{\left\|\frac{1}{T}\sum_{t=1}^{T}\left(\frac{\widehat{\boldsymbol\Lambda}'\boldsymbol\Lambda}{n}-\mathbf{J}\right)\frac{\Delta\mathbf{F}_t(\widehat{\mathbf{x}}_{t-1}-\mathbf{x}_{t-1})'\widehat{\boldsymbol\Lambda}\check{\boldsymbol\beta}}{n}\right\|}_{\mathcal{K}_2}$$

$$+\underbrace{\left\|\frac{1}{T}\sum_{t=1}^{T}\frac{\mathbf{F}_t(\Delta\widehat{\mathbf{x}}_{t-1}-\Delta\mathbf{x}_{t-1})'\widehat{\boldsymbol\Lambda}\check{\boldsymbol\beta}}{n}\left(\frac{\widehat{\boldsymbol\Lambda}'\boldsymbol\Lambda}{n}-\mathbf{J}\right)'\right\|}_{\mathcal{H}_2}+\underbrace{\left\|\frac{1}{T}\sum_{t=1}^{T}\frac{\widehat{\boldsymbol\Lambda}'\Delta\boldsymbol\xi_t(\widehat{\mathbf{x}}_{t-1}-\mathbf{x}_{t-1})'\widehat{\boldsymbol\Lambda}\check{\boldsymbol\beta}}{n^2}\right\|}_{\mathcal{L}_2}$$

$$+\underbrace{\left\|\frac{1}{T}\sum_{t=1}^{T}\frac{\widehat{\boldsymbol\Lambda}'\boldsymbol\xi_t(\Delta\widehat{\mathbf{x}}_{t-1}-\Delta\mathbf{x}_{t-1})'\widehat{\boldsymbol\Lambda}\check{\boldsymbol\beta}}{n^2}\right\|}_{\mathcal{M}_2}+\underbrace{\left\|\frac{1}{T}\sum_{t=1}^{T}\frac{\widehat{\boldsymbol\Lambda}'(\Delta\widehat{\mathbf{x}}_t-\Delta\mathbf{x}_t)(\widehat{\mathbf{x}}_{t-1}-\mathbf{x}_{t-1})'\widehat{\boldsymbol\Lambda}\check{\boldsymbol\beta}}{n^2}\right\|}_{\mathcal{N}_2}. \tag{D41}$$

Term $\mathcal{D}_2$ behaves like term $\mathcal{A}_2$, $\mathcal{E}_2$ and $\mathcal{G}_2$ behave like term $\mathcal{B}_2$, then since $\check{\boldsymbol\beta}\mathbf{J}\mathbf{F}_t'=\boldsymbol\beta'\mathbf{F}_t$ and therefore it is stationary, and because of because of (D35), $\mathcal{F}_2$ is $O_p(\max(T^{-1/2},n^{-1/2}))$ (this can be proved by simple modifications in the proof of Lemma D7). Terms $\mathcal{H}_2$, $\mathcal{K}_2$, and $\mathcal{N}_2$ are dominated by $\mathcal{C}_2$. Terms $\mathcal{L}_2$ and $\mathcal{M}_2$ behave as $\mathcal{C}_2$. We are left with term $\mathcal{J}_2$, which is such that

$$\begin{aligned}\mathcal{J}_2\leq&\left\|\frac{\widehat{\boldsymbol\Lambda}-\boldsymbol\Lambda\mathbf{J}}{\sqrt{n}}\right\|^2\left\|\frac{1}{T}\sum_{t=1}^{T}\frac{\Delta\boldsymbol\xi_t\boldsymbol\xi_{t-1}'}{n}\right\|\|\check{\boldsymbol\beta}\|+\left\|\frac{1}{T}\sum_{t=1}^{T}\frac{\boldsymbol\Lambda'\Delta\boldsymbol\xi_t\boldsymbol\xi_{t-1}'\boldsymbol\Lambda}{n^2}\right\|\|\check{\boldsymbol\beta}\|\\&+2\left\|\frac{\widehat{\boldsymbol\Lambda}-\boldsymbol\Lambda\mathbf{J}}{\sqrt{n}}\right\|\left\|\frac{1}{T}\sum_{t=1}^{T}\frac{\Delta\boldsymbol\xi_t\boldsymbol\xi_{t-1}'}{n}\right\|\left\|\frac{\boldsymbol\Lambda}{\sqrt{n}}\right\|\|\check{\boldsymbol\beta}\|=O_p\left(\frac{\sqrt{T}}{n^{2-\delta}}\right)+O_p\left(\frac{1}{n^{1-\delta}}\right),\end{aligned}$$

because of Lemma D7(vii) and D7(viii). Therefore, $\mathcal{J}_2$ is dominated by $\mathcal{B}_2$. By substituting (D40) and (D41) we have that (D39) is $O_p(\max(T^{1/2}n^{-(1-\delta/2)},T^{1/2}n^{-(1-\eta)},n^{(1-\delta)/2},n^{(1-\eta)/2},T^{-1/2}))$ and since $T^{1/2}n^{-(1-\delta/2)}<T^{1/2}n^{-(1-(\delta+\eta)/2)}$, then (D39) is also $O_p(\vartheta_{nT,\delta,\eta})$. Parts (vi), (vii), and (viii) are proved in the same way.

Last consider part (ix)

$$\begin{aligned}\left\|\frac{1}{T}\sum_{t=1}^{T}\check{\boldsymbol\beta}'\widehat{\mathbf{F}}_t\widehat{\mathbf{F}}_t'\check{\boldsymbol\beta}-\frac{1}{T}\sum_{t=1}^{T}\check{\boldsymbol\beta}'\check{\mathbf{F}}_t\check{\mathbf{F}}_t'\check{\boldsymbol\beta}\right\|\leq&\,2\left\|\frac{1}{T}\sum_{t=1}^{T}\check{\boldsymbol\beta}'\Big(\widehat{\mathbf{F}}_t-\mathbf{J}\mathbf{F}_t\Big)\Big(\check{\boldsymbol\beta}'\mathbf{J}\mathbf{F}_t\Big)'\right\|\\&+\left\|\frac{1}{T}\sum_{t=1}^{T}\check{\boldsymbol\beta}'\Big(\widehat{\mathbf{F}}_t-\mathbf{J}\mathbf{F}_t\Big)\Big(\widehat{\mathbf{F}}_t-\mathbf{J}\mathbf{F}_t\Big)'\check{\boldsymbol\beta}\right\|.\end{aligned} \tag{D42}$$

The first term on the rhs of (D42) behaves exactly as the first term on the rhs of (D39), so we just have to consider the second term on the rhs of (D42)

$$\left\|\frac{1}{T}\sum_{t=1}^{T}\check{\boldsymbol{\beta}}'\left(\widehat{\mathbf{F}}_t-\mathbf{J}\mathbf{F}_t\right)\left(\widehat{\mathbf{F}}_t-\mathbf{J}\mathbf{F}_t\right)'\check{\boldsymbol{\beta}}\right\|=\left\|\frac{1}{T}\sum_{t=1}^{T}\check{\boldsymbol{\beta}}'\left(\frac{\widehat{\boldsymbol{\Lambda}}'\widehat{\mathbf{x}}_t}{n}-\mathbf{J}\mathbf{F}_t\right)\left(\frac{\widehat{\boldsymbol{\Lambda}}'\widehat{\mathbf{x}}_t}{n}-\mathbf{J}\mathbf{F}_t\right)'\check{\boldsymbol{\beta}}\right\|$$
$$\le\underbrace{\left\|\frac{1}{T}\sum_{t=1}^{T}\frac{\check{\boldsymbol{\beta}}'\widehat{\boldsymbol{\Lambda}}'\boldsymbol{\Lambda}\mathbf{F}_t\mathbf{F}_t'}{n}\left(\frac{\boldsymbol{\Lambda}'\widehat{\boldsymbol{\Lambda}}}{n}-\mathbf{J}\right)\check{\boldsymbol{\beta}}+\check{\boldsymbol{\beta}}'\mathbf{J}\mathbf{F}_t\mathbf{F}_t'\left(\mathbf{J}-\frac{\boldsymbol{\Lambda}'\widehat{\boldsymbol{\Lambda}}}{n}\right)\check{\boldsymbol{\beta}}\right\|}_{\mathcal{A}_3}+2\underbrace{\left\|\frac{1}{T}\sum_{t=1}^{T}\frac{\check{\boldsymbol{\beta}}'\widehat{\boldsymbol{\Lambda}}'\boldsymbol{\Lambda}\mathbf{F}_t\boldsymbol{\xi}_t'\widehat{\boldsymbol{\Lambda}}\check{\boldsymbol{\beta}}}{n^2}\right\|}_{\mathcal{B}_3}$$
$$+2\underbrace{\left\|\frac{1}{T}\sum_{t=1}^{T}\frac{\check{\boldsymbol{\beta}}'\mathbf{J}\mathbf{F}_t\boldsymbol{\xi}_t'\widehat{\boldsymbol{\Lambda}}\check{\boldsymbol{\beta}}}{n}\right\|}_{\mathcal{C}_3}+\underbrace{\left\|\frac{1}{T}\sum_{t=1}^{T}\frac{\check{\boldsymbol{\beta}}'\widehat{\boldsymbol{\Lambda}}'\boldsymbol{\xi}_t\boldsymbol{\xi}_t'\widehat{\boldsymbol{\Lambda}}\check{\boldsymbol{\beta}}}{n^2}\right\|}_{\mathcal{D}_3}+2\underbrace{\left\|\frac{1}{T}\sum_{t=1}^{T}\check{\boldsymbol{\beta}}'\left(\frac{\widehat{\boldsymbol{\Lambda}}'\boldsymbol{\Lambda}}{n}-\mathbf{J}\right)\frac{\mathbf{F}_t(\widehat{\mathbf{x}}_t-\mathbf{x}_t)'\widehat{\boldsymbol{\Lambda}}\check{\boldsymbol{\beta}}}{n}\right\|}_{\mathcal{E}_3}$$
$$+2\underbrace{\left\|\frac{1}{T}\sum_{t=1}^{T}\frac{\check{\boldsymbol{\beta}}'\widehat{\boldsymbol{\Lambda}}'\boldsymbol{\xi}_t(\widehat{\mathbf{x}}_t-\mathbf{x}_t)'\widehat{\boldsymbol{\Lambda}}\check{\boldsymbol{\beta}}}{n^2}\right\|}_{\mathcal{F}_3}+\underbrace{\left\|\frac{1}{T}\sum_{t=1}^{T}\frac{\check{\boldsymbol{\beta}}'\widehat{\boldsymbol{\Lambda}}'(\widehat{\mathbf{x}}_t-\mathbf{x}_t)(\widehat{\mathbf{x}}_t-\mathbf{x}_t)'\widehat{\boldsymbol{\Lambda}}\check{\boldsymbol{\beta}}}{n^2}\right\|}_{\mathcal{G}_3}. \tag{D43}$$

Now term $\mathcal{A}_3$ is $O_p(T^{-1/2})$, because of (D35) and Lemma D6(v), terms $\mathcal{B}_3$ and $\mathcal{C}_3$ behave like term $\mathcal{B}_2$ in (D40), while term $\mathcal{E}_3$ is dominated by $\mathcal{C}_2$ in (D40). Then,

$$\mathcal{D}_3\le\left\|\frac{\widehat{\boldsymbol{\Lambda}}-\boldsymbol{\Lambda}\mathbf{J}}{\sqrt{n}}\right\|^2\left\|\frac{1}{T}\sum_{t=1}^{T}\frac{\boldsymbol{\xi}_t\boldsymbol{\xi}_t'}{n}\right\|\|\check{\boldsymbol{\beta}}\|^2+\left\|\frac{1}{T}\sum_{t=1}^{T}\frac{\boldsymbol{\Lambda}'\boldsymbol{\xi}_t\boldsymbol{\xi}_t'\boldsymbol{\Lambda}}{n^2}\right\|\|\check{\boldsymbol{\beta}}\|^2$$
$$+2\left\|\frac{\widehat{\boldsymbol{\Lambda}}-\boldsymbol{\Lambda}\mathbf{J}}{\sqrt{n}}\right\|\left\|\frac{1}{T}\sum_{t=1}^{T}\frac{\boldsymbol{\xi}_t\boldsymbol{\xi}_t'\boldsymbol{\Lambda}}{n^{3/2}}\right\|\|\check{\boldsymbol{\beta}}\|^2=O_p\left(\frac{T}{n^{2-\delta}}\right)+O_p\left(\frac{\sqrt{T}}{n^{(3-2\delta)/2}}\right)=O_p\left(\frac{\sqrt{T}}{n^{(2-\delta)/2}}\right),$$

because of Lemma D7(iii), D7(iv), and D7(ix) (multiplying the statements by T^2). Moreover,

$$\mathcal{F}_3\le\left\|\frac{\widehat{\boldsymbol{\Lambda}}-\boldsymbol{\Lambda}\mathbf{J}}{\sqrt{n}}\right\|^2\left\|\frac{1}{T}\sum_{t=1}^{T}\frac{\boldsymbol{\xi}_t(\widehat{\mathbf{x}}_t-\mathbf{x}_t)'}{n}\right\|\|\check{\boldsymbol{\beta}}\|^2+\left\|\frac{1}{T}\sum_{t=1}^{T}\frac{\boldsymbol{\Lambda}'\boldsymbol{\xi}_t(\widehat{\mathbf{x}}_t-\mathbf{x}_t)'\boldsymbol{\Lambda}}{n^2}\right\|\|\check{\boldsymbol{\beta}}\|^2$$
$$+2\left\|\frac{\widehat{\boldsymbol{\Lambda}}-\boldsymbol{\Lambda}\mathbf{J}}{\sqrt{n}}\right\|\left\|\frac{1}{T}\sum_{t=1}^{T}\frac{\boldsymbol{\xi}_t(\widehat{\mathbf{x}}_t-\mathbf{x}_t)'}{n}\right\|\left\|\frac{\boldsymbol{\Lambda}}{\sqrt{n}}\right\|\|\check{\boldsymbol{\beta}}\|^2=O_p\left(\frac{T}{n^{(2-\eta-\delta)}}\right)+O_p\left(\frac{\sqrt{T}}{n^{(2-\eta-\delta)/2}}\right),$$

because of Lemma D9(i) and D9(ii) (multiplying the statements by T^2). Last,

$$\mathcal{G}_3\le\left\|\frac{\widehat{\boldsymbol{\Lambda}}-\boldsymbol{\Lambda}\mathbf{J}}{\sqrt{n}}\right\|^2\left\|\frac{1}{T}\sum_{t=1}^{T}\frac{(\widehat{\mathbf{x}}_t-\mathbf{x}_t)(\widehat{\mathbf{x}}_t-\mathbf{x}_t)'}{n}\right\|\|\check{\boldsymbol{\beta}}\|^2+\left\|\frac{1}{T}\sum_{t=1}^{T}\frac{\boldsymbol{\Lambda}'(\widehat{\mathbf{x}}_t-\mathbf{x}_t)(\widehat{\mathbf{x}}_t-\mathbf{x}_t)'\boldsymbol{\Lambda}}{n^2}\right\|\|\check{\boldsymbol{\beta}}\|^2$$
$$+2\left\|\frac{\widehat{\boldsymbol{\Lambda}}-\boldsymbol{\Lambda}\mathbf{J}}{\sqrt{n}}\right\|\left\|\frac{1}{T}\sum_{t=1}^{T}\frac{(\widehat{\mathbf{x}}_t-\mathbf{x}_t)(\widehat{\mathbf{x}}_t-\mathbf{x}_t)'}{n}\right\|\left\|\frac{\boldsymbol{\Lambda}}{\sqrt{n}}\right\|\|\check{\boldsymbol{\beta}}\|^2=O_p\left(\frac{T}{n^{2(1-\eta)}}\right)+O_p\left(\frac{\sqrt{T}}{n^{1-\eta}}\right),$$

because of Lemma D8(iii) and D8(iv) (multiplying the statements by T^2).

By noticing that as $n,T\to\infty$, we have $\sqrt{T}n^{-(2-\eta-\delta)/2}\to 0$ (in $\mathcal{F}_3$) and $\sqrt{T}n^{-(1-\eta)}\to 0$ (in $\mathcal{G}_3$), we have

$$\mathcal{D}_3+\mathcal{F}_3+\mathcal{G}_3=O_p\left(\frac{\sqrt{T}}{n^{(2-\delta)/2}}\right)+O_p\left(\frac{\sqrt{T}}{n^{(2-\eta-\delta)/2}}\right)+O_p\left(\frac{\sqrt{T}}{n^{1-\eta}}\right). \tag{D44}$$

By substituting (D44) into (D43) and then (D43) into the second term on the rhs of (D42) and the results of part (v) for the second term on the rhs of (D42), we prove part (ix). This completes the proof.□

Lemma D11 *Define the matrices*

$$\widehat{\mathbf{S}}_{00} = \widehat{\mathbf{M}}_{00} - \widehat{\mathbf{M}}_{02}\widehat{\mathbf{M}}_{22}^{-1}\widehat{\mathbf{M}}_{20}, \qquad \widehat{\mathbf{S}}_{01} = \widehat{\mathbf{M}}_{01} - \widehat{\mathbf{M}}_{02}\widehat{\mathbf{M}}_{22}^{-1}\widehat{\mathbf{M}}_{21}, \qquad \widehat{\mathbf{S}}_{11} = \widehat{\mathbf{M}}_{11} - \widehat{\mathbf{M}}_{12}\widehat{\mathbf{M}}_{22}^{-1}\widehat{\mathbf{M}}_{21},$$

where $\widehat{\mathbf{M}}_{10} = \widehat{\mathbf{M}}'_{01}$, $\widehat{\mathbf{M}}_{20} = \widehat{\mathbf{M}}'_{02}$, *and* $\widehat{\mathbf{M}}_{12} = \widehat{\mathbf{M}}'_{21}$. *Denote by* $\mathbf{S}_{ij}$, *for* $i,j = 0,1$, *the analogous ones but computed by using* $\check{\mathbf{F}}_t = \mathbf{J}\mathbf{F}_t$. *Define also* $\check{\boldsymbol{\beta}} = \mathbf{J}\boldsymbol{\beta}$ *and* $\check{\boldsymbol{\beta}}_{\perp *} = \check{\boldsymbol{\beta}}_\perp(\check{\boldsymbol{\beta}}'_\perp\check{\boldsymbol{\beta}}_\perp)^{-1}$, *where* $\check{\boldsymbol{\beta}}_\perp = \mathbf{J}\boldsymbol{\beta}_\perp$ *such that* $\check{\boldsymbol{\beta}}'_\perp\check{\boldsymbol{\beta}} = \mathbf{0}_{r-c\times r}$. *Under Assumptions 1 through 5, as* $n, T \to \infty$,

(i) $\|\widehat{\mathbf{S}}_{00} - \mathbf{S}_{00}\| = O_p(\max(n^{-1/2}, T^{-1/2}, n^{-(1-\eta)}))$.

If also Assumption 6 holds, then,

(ii) $\|\check{\boldsymbol{\beta}}'\widehat{\mathbf{S}}_{11}\check{\boldsymbol{\beta}} - \check{\boldsymbol{\beta}}'\mathbf{S}_{11}\check{\boldsymbol{\beta}}\| = O_p(\vartheta_{nT,\delta,\eta})$;

(iii) $\|T^{-1/2}\check{\boldsymbol{\beta}}'\widehat{\mathbf{S}}_{11}\check{\boldsymbol{\beta}}_{\perp *} - T^{-1/2}\check{\boldsymbol{\beta}}'\mathbf{S}_{11}\check{\boldsymbol{\beta}}_{\perp *}\| = O_p(\vartheta_{nT,\delta,\eta})$;

(iv) $\|T^{-1/2}\check{\boldsymbol{\beta}}'\widehat{\mathbf{S}}_{10}\widehat{\mathbf{S}}_{00}^{-1}\widehat{\mathbf{S}}_{01}\check{\boldsymbol{\beta}}_{\perp *} - T^{-1/2}\check{\boldsymbol{\beta}}'\mathbf{S}_{10}\mathbf{S}_{00}^{-1}\mathbf{S}_{01}\check{\boldsymbol{\beta}}_{\perp *}\| = O_p(\vartheta_{nT,\delta,\eta})$;

(v) $\|T^{-1}\check{\boldsymbol{\beta}}'_{\perp *}\widehat{\mathbf{S}}_{10}\widehat{\mathbf{S}}_{00}^{-1}\widehat{\mathbf{S}}_{01}\check{\boldsymbol{\beta}}_{\perp *} - T^{-1}\check{\boldsymbol{\beta}}'_{\perp *}\mathbf{S}_{10}\mathbf{S}_{00}^{-1}\mathbf{S}_{01}\check{\boldsymbol{\beta}}_{\perp *}\| = O_p(\vartheta_{nT,\delta,\eta})$;

(vi) $\|T^{-1}\check{\boldsymbol{\beta}}'_{\perp *}\widehat{\mathbf{S}}_{11}\check{\boldsymbol{\beta}}_{\perp *} - T^{-1}\check{\boldsymbol{\beta}}'_{\perp *}\mathbf{S}_{11}\check{\boldsymbol{\beta}}_{\perp *}\| = O_p(\vartheta_{nT,\delta,\eta})$.

Proof. Throughout we use the fact that $\|\check{\boldsymbol{\beta}}_{\perp *}\| = O(1)$. Part (i) is proved using Lemma D10(ii), D10(iii) and D10(iv). For proving part (ii) we use Lemma D10(iv), D10(vi) and D10(ix). Part (iii) is proved by combining part (ii) with Lemma D10(v) and D10(ix), and by noticing that $\|T^{-1/2}\mathbf{F}_t\| = O_p(1)$ from Lemma D4(ii). For proving part (iv) we combine part (i) with Lemma D10(v), D10(vii) and D10(viii). Part (v) is proved by combining part (i) with Lemma D10(vii) and D10(viii). Finally, part (vi) follows from Lemma D10(i) and D10(viii). This completes the proof. □

Lemma D12 *Consider the matrices* $\mathbf{S}_{ij}$ *defined in Lemma D11, with* $i,j = 0,1$. *Define* $\check{\mathbf{F}}_t = \mathbf{J}\mathbf{F}_t$, $\check{\boldsymbol{\beta}} = \mathbf{J}\boldsymbol{\beta}$ *and the conditional covariance matrices*

$$\check{\boldsymbol{\Omega}}_{00} = \mathsf{E}[\Delta\check{\mathbf{F}}_t\Delta\check{\mathbf{F}}'_t|\Delta\check{\mathbf{F}}_{t-1}], \quad \check{\boldsymbol{\Omega}}_{\check{\beta}\check{\beta}} = \mathsf{E}[\check{\boldsymbol{\beta}}'\check{\mathbf{F}}_{t-1}\check{\mathbf{F}}'_{t-1}\check{\boldsymbol{\beta}}|\Delta\check{\mathbf{F}}_{t-1}], \quad \check{\boldsymbol{\Omega}}_{0\check{\beta}} = \mathsf{E}[\Delta\check{\mathbf{F}}_t\check{\mathbf{F}}'_{t-1}\check{\boldsymbol{\beta}}|\Delta\check{\mathbf{F}}_{t-1}].$$

Under Assumptions 1 and 4, as $T \to \infty$,

(i) $\|\mathbf{S}_{00} - \check{\boldsymbol{\Omega}}_{00}\| = O_p(T^{-1/2})$;

(ii) $\|\check{\boldsymbol{\beta}}'\mathbf{S}_{11}\check{\boldsymbol{\beta}} - \check{\boldsymbol{\Omega}}_{\check{\beta}\check{\beta}}\| = O_p(T^{-1/2})$;

(iii) $\|\mathbf{S}_{01}\check{\boldsymbol{\beta}} - \check{\boldsymbol{\Omega}}_{0\check{\beta}}\| = O_p(T^{-1/2})$.

Proof. For part (i), notice that

$$\check{\boldsymbol{\Omega}}_{00} = \mathsf{E}[\Delta\check{\mathbf{F}}_t\Delta\check{\mathbf{F}}'_t] - \mathsf{E}[\Delta\check{\mathbf{F}}_t\Delta\check{\mathbf{F}}'_{t-1}]\left(\mathsf{E}[\Delta\check{\mathbf{F}}_{t-1}\Delta\check{\mathbf{F}}'_{t-1}]\right)^{-1}\mathsf{E}[\Delta\check{\mathbf{F}}_{t-1}\Delta\check{\mathbf{F}}'_t] = \boldsymbol{\Gamma}_0^{\Delta F} - \boldsymbol{\Gamma}_1^{\Delta F}\left(\boldsymbol{\Gamma}_0^{\Delta F}\right)^{-1}\boldsymbol{\Gamma}_1^{\Delta F},$$

and

$$\begin{aligned}\mathbf{S}_{00} &= \frac{1}{T}\sum_{t=1}^T \Delta\check{\mathbf{F}}_t\Delta\check{\mathbf{F}}'_t - \left(\frac{1}{T}\sum_{t=2}^T \Delta\check{\mathbf{F}}_t\Delta\check{\mathbf{F}}'_{t-1}\right)\left(\frac{1}{T}\sum_{t=2}^T \Delta\check{\mathbf{F}}_{t-1}\Delta\check{\mathbf{F}}'_{t-1}\right)^{-1}\frac{1}{T}\sum_{t=2}^T \Delta\check{\mathbf{F}}_{t-1}\Delta\check{\mathbf{F}}'_t \\ &= \mathbf{M}_{00} - \mathbf{M}_{02}\mathbf{M}_{22}^{-1}\mathbf{M}_{20}.\end{aligned}$$

Using Lemma D6(i), we have the result. Parts (ii) and (iii) are proved in the same way, but using Lemma D6(v) and D6(vi), respectively. This completes the proof. □

Lemma D13 *Under Assumptions 1 through 3, there exist positive reals* $\underline{M}_9$, $\overline{M}_9$, M_{10}, $\underline{M}_{11}$, $\overline{M}_{11}$ *and an integer* $\bar{n}$ *such that*

(i) $\underline{M}_9 \le n^{-1}\nu_j^{\Delta\chi}(\theta) \le \overline{M}_9$ *a.e. in* $[-\pi,\pi]$, *and for any* $j = 1,\dots,q$ *and* $n > \bar{n}$;

(ii) $\sup_{\theta\in[-\pi,\pi]} \nu_1^{\Delta\xi}(\theta) \le M_{10}$, *for any* $n \in \mathbb{N}$;

(iii) $\underline{M}_{11} \le n^{-1}\nu_j^{\Delta x}(\theta) \le \overline{M}_{11}$ *a.e. in* $[-\pi,\pi]$, *and for any* $j = 1,\dots,q$ *and* $n > \bar{n}$;

(iv) $\sup_{\theta\in[-\pi,\pi]}\nu_{q+1}^{\Delta x}(\theta)\le M_{10}$, *for any* $n\in\mathbb{N}$;

(v) $\underline{M}_{12}\le n^{-1}\nu_j^{\Delta x}(0)\le \overline{M}_{12}$, *for any* $j=1,\dots,\tau$ *and* $n>\bar{n}$;

(vi) $\nu_{\tau+1}^{\Delta x}(0)\le M_{10}$, *for any* $n\in\mathbb{N}$.

Proof. For part (i) we can follow a reasoning similar to Lemma D2(i). The spectral density matrix of the first difference of the common factors can be written as $\mathbf{\Sigma}^{\Delta F}(\theta)=(2\pi)^{-1}\mathbf{C}(e^{-i\theta})\overline{\mathbf{C}'(e^{-i\theta})}$ and, since $\mathrm{rk}(\mathbf{C}(e^{-i\theta}))=q$ a.e. in $[-\pi,\pi]$, then it has q non-zero real eigenvalues and $r-q$ zero eigenvalues. Notice also that we have $\mathrm{rk}(\mathbf{C}(e^{-i\theta}))\le q$ for any $\theta\in[-\pi,\pi]$. Moreover, given square summability of the coefficients of $\mathbf{C}(L)$ as a consequence of Assumption 1(b), the non-zero eigenvalues are also finite for any $\theta\in[-\pi,\pi]$. Thus, by denoting as $\nu_j^{\Delta F}(\theta)$ such eigenvalues, there exist positive reals $\underline{M}_{10}$ and $\overline{M}_{10}$ such that a.e. in $[-\pi,\pi]$

$$\underline{M}_{10}\le \nu_j^{\Delta F}(\theta)\le \overline{M}_{10},\quad j=1,\dots,q. \tag{D45}$$

Therefore, we can write $\mathbf{\Sigma}^{\Delta F}(\theta)=\mathbf{W}^{\Delta F}(\theta)\mathbf{M}^{\Delta F}(\theta)\overline{\mathbf{W}^{\Delta F'}(\theta)}$, where $\mathbf{W}^{\Delta F}(\theta)$ is the $r\times q$ matrix of normalised eigenvectors, i.e. such that $\overline{\mathbf{W}^{\Delta F'}(\theta)}\mathbf{W}^{\Delta F}(\theta)=\mathbf{I}_q$ for any $\theta\in[-\pi,\pi]$, and $\mathbf{M}^{\Delta F}(\theta)$ is the corresponding $q\times q$ diagonal matrix of eigenvalues.

Define $\boldsymbol{L}(\theta)=\mathbf{\Lambda}\mathbf{W}^{\Delta F}(\theta)(\mathbf{M}^{\Delta F}(\theta))^{1/2}$ for any $\theta\in[-\pi,\pi]$. Then the spectral density matrix of the first differences of the common component is given by

$$\frac{\mathbf{\Sigma}^{\Delta\chi}(\theta)}{n}=\frac{1}{n}\mathbf{\Lambda}\mathbf{\Sigma}^{\Delta F}(\theta)\mathbf{\Lambda}'=\frac{1}{n}\mathbf{\Lambda}\mathbf{W}^{\Delta F}(\theta)\mathbf{M}^{\Delta F}(\theta)\overline{\mathbf{W}^{\Delta F'}(\theta)}\mathbf{\Lambda}'=\frac{\boldsymbol{L}(\theta)\overline{\boldsymbol{L}'(\theta)}}{n},\quad \theta\in[-\pi,\pi].$$

Moreover, since because of Assumption 2(a), there exists an integer $\bar{n}$ such that $n^{-1}\mathbf{\Lambda}'\mathbf{\Lambda}=\mathbf{I}_r$, for any $n>\bar{n}$, then

$$\frac{\overline{\boldsymbol{L}'(\theta)}\boldsymbol{L}(\theta)}{n}=\mathbf{M}^{\Delta F}(\theta),\quad \theta\in[-\pi,\pi]. \tag{D46}$$

Therefore, a.e. in $[-\pi,\pi]$ the non-zero dynamic eigenvalues of $\mathbf{\Sigma}^{\Delta\chi}(\theta)$ are the same as those of $\overline{\boldsymbol{L}'(\theta)}\boldsymbol{L}(\theta)$, and from (D46), we have for any $n>\bar{n}$ and a.e. in $[-\pi,\pi]$, $n^{-1}\nu_j^{\Delta\chi}(\theta)=\nu_j^{\Delta F}(\theta)$, for any $j=1,\dots,r$. Part (i) then follows from (D45).

As for part (ii), from Assumption 3(c), for any $\theta\in[-\pi,\pi]$, there exists a positive real M_4 such that

$$\sup_{i\in\mathbb{N}}\left|\check{d}_i(e^{-i\theta})\right|\le\sup_{i\in\mathbb{N}}\left|\sum_{k=0}^{\infty}\check{d}_{ik}e^{-ik\theta}\right|\le\sup_{i\in\mathbb{N}}\sum_{k=0}^{\infty}\left|\check{d}_{ik}\right|\le M_4. \tag{D47}$$

Define as $\sigma_{ij}(\theta)$ the generic (i,j)-th entry of $\mathbf{\Sigma}^{\Delta\xi}(\theta)$. Then, for any $n>\bar{n}$,

$$\begin{aligned}\sup_{\theta\in[-\pi,\pi]}\left\|\mathbf{\Sigma}^{\Delta\xi}(\theta)\right\|_1&=\sup_{\theta\in[-\pi,\pi]}\max_{i=1,\dots,n}\sum_{j=1}^{n}|\sigma_{ij}(\theta)|=\sup_{\theta\in[-\pi,\pi]}\max_{i=1,\dots,n}\frac{1}{2\pi}\sum_{j=1}^{n}\left|\check{d}_i(e^{-i\theta})\mathsf{E}[\varepsilon_{it}\varepsilon_{jt}]\,\check{d}_j(e^{i\theta})\right|\\&\le\frac{M_4^2}{2\pi}\max_{i=1,\dots,n}\sum_{j=1}^{n}|\mathsf{E}[\varepsilon_{it}\varepsilon_{jt}]|\le\frac{M_4^2M_3}{2\pi},\end{aligned} \tag{D48}$$

where we used (D47) and Assumption 3(b). From (C2) and (D48), we have, for any $n>\bar{n}$,

$$\sup_{\theta\in[-\pi,\pi]}\nu_1^{\Delta\xi}(\theta)=\sup_{\theta\in[-\pi,\pi]}\left\|\mathbf{\Sigma}^{\Delta\xi}(\theta)\right\|\le\sup_{\theta\in[-\pi,\pi]}\left\|\mathbf{\Sigma}^{\Delta\xi}(\theta)\right\|_1\le\frac{M_4^2M_3}{2\pi}, \tag{D49}$$

and part (ii) is proved by defining $M_{11}=M_4^2M_3(2\pi)^{-1}$.

Finally, parts (iii) and (iv), are immediate consequences of Assumption 3(e), which implies that $\mathbf{\Sigma}^{\Delta x}(\theta)=\mathbf{\Sigma}^{\Delta\chi}(\theta)+\mathbf{\Sigma}^{\Delta\xi}(\theta)$, for any $\theta\in[-\pi,\pi]$, and of Weyl's inequality (C3). So, for $j=1,\dots,q$, and for any $n > \bar{n}$ and a.e. in $[-\pi, \pi]$, there exist positive reals $\underline{M}_{12}$ and $\overline{M}_{12}$ such that

$$\frac{\nu_j^{\Delta x}(\theta)}{n} \le \frac{\nu_j^{\Delta \chi}(\theta)}{n} + \frac{\nu_1^{\Delta \xi}(\theta)}{n} \le \overline{M}_{10} + \sup_{\theta \in [-\pi,\pi]} \frac{\nu_1^{\Delta \xi}(\theta)}{n} \le \overline{M}_{10} + \frac{M_{11}}{n} = \overline{M}_{12},$$

$$\frac{\nu_j^{\Delta x}(\theta)}{n} \ge \frac{\nu_j^{\Delta \chi}(\theta)}{n} + \frac{\nu_n^{\Delta \xi}(\theta)}{n} \ge \underline{M}_{10} + \inf_{\theta \in [-\pi,\pi]} \frac{\nu_n^{\Delta \xi}(\theta)}{n} = \underline{M}_{12}.$$

because of parts (i) and (ii). This proves part (iii). When $j = q + 1$, using parts (i) and (ii), and since $\text{rk}(\mathbf{\Sigma}^{\Delta \chi}(\theta)) \le q$, for any $\theta \in [-\pi, \pi]$, we have $\nu_{q+1}^{\Delta x}(\theta) \le \nu_{q+1}^{\Delta \chi}(\theta) + \nu_1^{\Delta \xi(\theta)} = \nu_1^{\Delta \xi(\theta)} \le M_{11}$, thus proving part (iv).

Finally, for parts (v) and (vi) consider parts (iii) and (iv) but when $\theta = 0$. Then, $\text{rk}(\mathbf{\Sigma}^{\Delta \chi}(0)) = \tau \le q$ which implies $\underline{M}_{10} \le n^{-1}\nu_\tau^{\Delta \chi}(0) \le \overline{M}_{10}$, but $\nu_{\tau+1}^{\Delta \chi}(0) = 0$. Using again parts (i) and (ii) and Weyl's inequality (C3), we prove parts (v) and (vi). This completes the proof. □

E Details on identification of IRFs and their confidence bands

E1 Identification

As we discuss in Section 3.2, the IRFs in (21) are in general not identified unless we also estimate the orthogonal $q \times q$ transformation $\mathbf{R}$. Economic theory tells us that the choice of the identifying transformation can be determined by the economic meaning attached to the common shocks, $\mathbf{u}_t$. In general, for a given set of identifying restrictions, $\mathbf{R}$ depends on the other parameters of the model, that is, it is determined by a mapping $\mathbf{R} \equiv \mathbf{R}(\mathbf{\Lambda}, \mathbf{A}(L), \mathbf{K})$. In the typical case of just- or under-identifying restrictions, to estimate $\mathbf{R}$ we just have to consider the q rows of the raw estimated IRFs, denoted as $\widetilde{\mathbf{\Phi}}_{[q]}(L)$, corresponding to the economic variables which are relevant for identification of the shocks. Therefore, we define the estimator $\widehat{\mathbf{R}}$ such that $\widetilde{\mathbf{\Phi}}_{[q]}(L)\widehat{\mathbf{R}}$ satisfies our desired restrictions. In this case, due to orthogonality, an estimator $\widehat{\mathbf{R}}$ is obtained by solving a linear system of $q(q-1)/2$ equations with $q(q-1)/2$ unknowns, which depends on $\widetilde{\mathbf{\Phi}}_{[q]}(L)$ and therefore on $\widehat{\mathbf{\Lambda}}$, $\widehat{\mathbf{A}}^{\text{VECM}}(L)$, and $\widehat{\mathbf{K}}$. Once we have computed $\widehat{\mathbf{R}}$, the $n \times q$ matrix of identified IRFs is $\widehat{\mathbf{\Phi}}(L) = \widetilde{\mathbf{\Phi}}(L)\widehat{\mathbf{R}}$. Finally, if we denote the raw shocks as $\widetilde{\mathbf{u}}_t$, the identified shocks are given by $\widehat{\mathbf{u}}_t = \widehat{\mathbf{R}}'\widetilde{\mathbf{u}}_t$. Details on the two identification schemes adopted in Section 6 are given below.

Application 1: Oil price shock. To identify the oil price shock, Stock and Watson (2016) use a standard recursive identification scheme such that an oil price shock is the only shock having contemporaneous effect on the oil price. Specifically, when $q = 3$, let x_{1t} be the oil price, x_{2t} be GDP, and x_{3t} be consumption; then, $\widehat{\mathbf{R}}$ must be such that $\widehat{\mathbf{\Phi}}_{[3]}(0) = \widetilde{\mathbf{\Phi}}_{[3]}(0)\widehat{\mathbf{R}}$ is lower triangular, i.e. such that the identified IRFs are given by

$$\widehat{\mathbf{\Phi}}_{[3]}(0) = \begin{bmatrix} \widehat{\phi}_{11}(0) & 0 & 0 \\ \widehat{\phi}_{21}(0) & \widehat{\phi}_{22}(0) & 0 \\ \widehat{\phi}_{31}(0) & \widehat{\phi}_{32}(0) & \widehat{\phi}_{33}(0) \end{bmatrix} = \widetilde{\mathbf{\Phi}}_{[3]}(0)\widehat{\mathbf{R}}.$$

Therefore, we can choose $\widehat{\mathbf{R}} = [\widetilde{\mathbf{\Phi}}_{[3]}(0)]^{-1}\widetilde{\mathbf{R}}$, where $\widetilde{\mathbf{R}}$ is the lower triangular Choleski factor such that $\widetilde{\mathbf{\Phi}}_{[3]}(0)\widetilde{\mathbf{\Phi}}_{[3]}(0)' = \widetilde{\mathbf{R}}\widetilde{\mathbf{R}}'$. The oil price shock is then obtained as $\widehat{u}_{1t} = \widehat{\mathbf{r}}_1'\widetilde{\mathbf{u}}_t$, where $\widehat{\mathbf{r}}_1$ is the first column of $\widehat{\mathbf{R}}$. The identified IRFs, reported in Figure 1, are given by the entries of the first column of $\widehat{\mathbf{\Phi}}(L)$, corresponding to the variables considered.

Application 2: News shock. To identify the news shock, Forni et al. (2014) proceed as follows: first, they identify what they call a "surprise technology shock" as the only shock having a contemporaneous effect on TFP; next, they identify the news shock by imposing that out of the remaining four shocks, the news shock is the one with maximal impact on TFP at lag 60. In practice, this identification is obtained as follows—recall that the considered FAVAR is composed of two variables (TFP and stock prices) and three estimated factors so that $q = 5$: Let x_{1t} and x_{2t} be TFP and stock prices, respectively, and let x_{3t}, x_{4t}, x_{5t} be GDP, consumption, and investment.

(a) The surprise technology shock is identified by setting $\widehat{\mathbf{R}}$ such that $\widehat{\mathbf{\Phi}}_{[5]}(0) = \widetilde{\mathbf{\Phi}}_{[5]}(0)\widehat{\mathbf{R}}$ is lower triangular, i.e. such that the identified IRFs are given by

$$\widehat{\mathbf{\Phi}}_{[5]}(0) = \begin{bmatrix} \widehat{\phi}_{11}(0) & 0 & 0 & 0 & 0 \\ \widehat{\phi}_{21}(0) & \widehat{\phi}_{22}(0) & 0 & 0 & 0 \\ \widehat{\phi}_{31}(0) & \widehat{\phi}_{32}(0) & \widehat{\phi}_{33}(0) & 0 & 0 \\ \widehat{\phi}_{41}(0) & \widehat{\phi}_{42}(0) & \widehat{\phi}_{43}(0) & \widehat{\phi}_{44}(0) & 0 \\ \widehat{\phi}_{51}(0) & \widehat{\phi}_{52}(0) & \widehat{\phi}_{53}(0) & \widehat{\phi}_{54}(0) & \widehat{\phi}_{55}(0) \end{bmatrix} = \widetilde{\mathbf{\Phi}}_{[5]}(0)\widehat{\mathbf{R}}.$$

Therefore, we can choose $\widehat{\mathbf{R}} = [\widetilde{\mathbf{\Phi}}_{[5]}(0)]^{-1}\widetilde{\mathbf{R}}$, where $\widetilde{\mathbf{R}}$ is the lower triangular Choleski factor such that $\widetilde{\mathbf{\Phi}}_{[5]}(0)\widetilde{\mathbf{\Phi}}_{[5]}(0)' = \widetilde{\mathbf{R}}\widetilde{\mathbf{R}}'$.

(b) The news shock is then identified by choosing the 4×1 vector $\widehat{\mathbf{r}}_2 = (0\, \widehat{r}_{22}\, \widehat{r}_{32}\, \widehat{r}_{42}\, \widehat{r}_{52})'$ such that $\widehat{\mathbf{r}}_2'\widehat{\mathbf{r}}_2 = 1$ and it maximizes the element $(1,1)$ of $\widehat{\mathbf{\Phi}}_{[5]}(60) = \widetilde{\mathbf{\Phi}}_{[5]}(60)\widehat{\mathbf{r}}_2$, which is the effect of the news shock on TFP at lag 60. The news shock is then obtained as $\widehat{u}_{2t} = \widehat{\mathbf{r}}_2'\widetilde{\mathbf{u}}_t$. The identified IRFs to a news shock, reported in Figure 2, are given by the entries of the second column of $\widehat{\mathbf{\Phi}}(L)$, corresponding to the variables considered.

E2 Bootstrap confidence bands in practice

In order to build confidence intervals for the estimated IRFs, we use a bootstrap algorithm. In detail, at each iteration $d = 1, \ldots, 1000$, we generate bootstrap shocks $\mathbf{u}_t^d$ by drawing randomly with replacement from the estimated shocks $\widehat{\mathbf{u}}_t$ and we generate bootstrap common factors $\mathbf{F}_t^d$. Then, we estimate $\widehat{\mathbf{A}}(L)^d$, $\widehat{\mathbf{K}}^d$, and $\widehat{\mathbf{R}}^d$ in (22) or (26), thus obtaining a bootstrap IRF $\widehat{\mathbf{\Phi}}(L)^d = \widehat{\mathbf{\Lambda}}[\widehat{\mathbf{A}}(L)^d]^{-1}\widehat{\mathbf{K}}^d\widehat{\mathbf{R}}^d$. Repeating this procedure several times gives, for each i, j and lag k, a bootstrap distribution of the IRF: $\{\widehat{\phi}_{ij,k}^d,\ d = 1, \ldots 1000\}$ (for simplicity below we omit the dependence on i and j of the IRF).

In order to compute the $(1-\alpha)$ confidence interval, at each lag k we compute the sample variance of $\{\widehat{\phi}_k^d\}$, which we denote as σ_k^2, and then we construct the $(1-\alpha)$ confidence interval is given by $[\widehat{\phi}_k + z_{\alpha/2}\,\sigma_k,\ \widehat{\phi}_k + z_{1-\alpha/2}\,\sigma_k]$, where $z_{\alpha/2} = -z_{1-\alpha/2}$ is the $\alpha/2$ quantile of a standard normal, see also Chapter 12 in Kilian and Lütkepohl (2017). By proceeding in this way we obtain symmetric confidence bands around the estimated IRF.

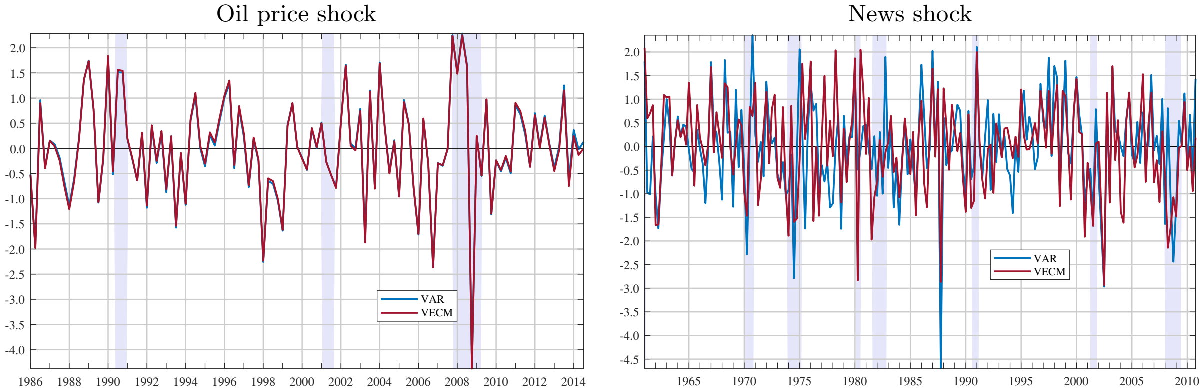
Stock and Watson (2016) adopt a procedure very similar to the one described above. By contrast, Forni et al. (2014) compute the confidence bands as the percentiles of $\{\widehat{\phi}_k^d\}$ over the replication d. This is also a a possible strategy, which yields confidence bands that are not symmetrical by construction, but does not ensure that the estimated IRF is within the confidence bands.

E3 Estimated identified shocks

In Section 6, we show and discuss the estimated IRFs, which are our main object of interest. In contrast, we said nothing about the identified shocks, which, although they are not the object of interest in the empirical application, they are intimately intertwined with the IRFs, as we explain in Section E1.

Figure 1 shows the estimated shocks. The left plot reports the oil price shock identified as in Stock and Watson (2016), while the right plot reports the news shock identified as in Forni et al. (2014). The figure shows both the estimate obtained by estimating an unrestricted VAR on $\widehat{\mathbf{F}}_t$ or a VECM on $\Delta\widehat{\mathbf{F}}_t$. As we can see, the two estimates of the oil price shock are nearly indistinguishable, which dovetail with the estimated IRFs shown in Figure 1 in the paper. By contrast, the news shock differs depending on which law of motion is estimated for the common factors, which, as we explained in Section 3 in the paper, depends on the fact that the restriction is imposed at lag 60, and therefore it depends on the estimated of the long-run IRFs.

Figure 1: ESTIMATED IDENTIFIED SHOCKS



In each plot, the red line is the shock estimated by fitting a VECM on $\Delta\widehat{\mathbf{F}}_t$, while the blue line is the shock estimated by fitting a VAR on $\widehat{\mathbf{F}}_t$. In the left plot, the two estimated shocks are so similar that the red line overlap completely the blue line.

F Factor Augment VAR models

F1 On the relation between FAVAR and DFM

Consider the FAVAR model proposed by Bernanke et al. (2005):

$$\mathbf{w}_t = \boldsymbol{L}^f \mathbf{f}_t + \boldsymbol{L}^z \mathbf{z}_t + \mathbf{e}_t, \qquad \boldsymbol{\Psi}(L)\begin{bmatrix}\mathbf{f}_t\\ \mathbf{z}_t\end{bmatrix} = \mathbf{v}_t, \tag{F1}$$

where $\mathbf{z}_t$ is an m-dimensional vector of observable economic variables of interest, $\mathbf{f}_t$ is a k-dimensional vector of latent factors summarising additional information contained in the N-dimensional vector $\mathbf{w}_t$. In this setting $\mathbf{e}_t$ is the idiosyncratic component of $\mathbf{w}_t$ and $\mathbf{v}_t$ is a white noise process containing the structural shocks that we are interested in and it is of dimension $k+m \ll N$.

Following Stock and Watson (2016, Section 5.2), let

$$\mathbf{x}_t = \begin{bmatrix}\mathbf{w}_t\\ \mathbf{z}_t\end{bmatrix} \qquad \text{and} \qquad \bar{\mathbf{F}}_t = \begin{bmatrix}\mathbf{f}_t\\ \mathbf{z}_t\end{bmatrix},$$

where $\mathbf{x}_t$ is the vector of all observed time series of dimension $n = N+m$ and $\bar{\mathbf{F}}_t$ is $(m+k)$-dimensional. Then, we can rewrite (F1) as:

$$\mathbf{x}_t = \bar{\boldsymbol{\Lambda}}\bar{\mathbf{F}}_t + \bar{\boldsymbol{\xi}}_t, \qquad \boldsymbol{\Psi}(L)\bar{\mathbf{F}}_t = \mathbf{v}_t, \tag{F2}$$

where:

$$\bar{\boldsymbol{\Lambda}} = \begin{bmatrix}\boldsymbol{L}^f & \boldsymbol{L}^z\\ \mathbf{0}_{m\times r} & \mathbf{I}_m\end{bmatrix} \qquad \text{and} \qquad \bar{\boldsymbol{\xi}}_t = \begin{bmatrix}\mathbf{e}_t\\ \mathbf{0}_{m\times 1}\end{bmatrix}.$$

On the other hand the DFM reads

$$\mathbf{x}_t = \boldsymbol{\Lambda}\mathbf{F}_t + \boldsymbol{\xi}_t, \qquad \mathbf{A}(L)\mathbf{F}_t = \mathbf{K}\mathbf{u}_t. \tag{F3}$$

Therefore, the FAVAR (F2) is a restricted version of the DFM (F3), where the variables $\mathbf{z}_t$ have unit factor loadings and zero idiosyncratic component and the number of factors is $r = k+m$, which is equal to the number of common shocks, i.e. in (F3) we also impose $r = q$ and thus $\mathbf{K} = \mathbf{I}_r$. In other words in a FAVAR the variables of interest $\mathbf{z}_t$ are considered as "observable" factors. Although the FAVAR has been mainly studied in a stationary setting, the same reasoning applies if we have non-stationary data. Note that deterministic linear trends can also be included in the FAVAR as we discuss in the next section.

F2 FAVAR estimation

Let y_{it} be the observed data, then in our framework the FAVAR is written as

$$\begin{aligned} y_{it} &= a_i + b_i t + x_{it}, \\ x_{it} &= (\boldsymbol{l}_i^{f'} \quad \boldsymbol{l}_i^{z'})(\mathbf{f}_t' \quad \mathbf{z}_t')' + \xi_{it}, \\ \boldsymbol{\Psi}(L)(\mathbf{f}_t' \quad \mathbf{z}_t')' &= \mathbf{v}_t, \end{aligned}$$

where $\mathbf{z}_t$ are the "observed" common factors, and $\mathbf{f}_t$ are the "unobserved" common factors. The model is estimated as follows:

1. estimate the unobserved common factors $\widehat{\mathbf{f}}_t$ from $\mathbf{y}_t = (y_{1t} \cdots y_{nt})'$ as explained in Section 3, thus de-trending series first (if needed);
2. estimate $\widehat{\boldsymbol{\Psi}}(L)$ by fitting either a VECM on $(\Delta\widehat{\mathbf{f}}_t' \quad \Delta\mathbf{z}_t')'$ or an unrestricted VAR on $(\widehat{\mathbf{f}}_t' \quad \mathbf{z}_t')'$ as explained in Section 3;
3. estimate $(\widehat{\boldsymbol{l}}_i^{f'} \quad \widehat{\boldsymbol{l}}_i^{z'})$ by regressing Δy_{it} onto a constant and the vector $(\Delta\mathbf{f}_t' \quad \Delta\widehat{\mathbf{z}}_t')'$;
4. estimate IRFs as $(\widehat{\boldsymbol{l}}_i^{f'} \quad \widehat{\boldsymbol{l}}_i^{z'})[\widehat{\boldsymbol{\Psi}}(L)]^{-1}$.

In contrast, in the approach by Forni et al. (2014) the factors are extracted directly from the observed data y_{it}, without controlling for the presence of possible deterministic linear trends. Therefore, the FAVAR is written as

$$y_{it} = (\boldsymbol{l}_i^{f'} \quad \boldsymbol{l}_i^{z'})(\mathbf{f}_t' \quad \mathbf{z}_t')' + \xi_{it}.$$

The model is estimated as follows:

1. estimate the unobserved common factors from PC analysis of $\mathbf{y}_t = (y_{1t} \cdots y_{nt})'$ as in Bai (2004);
2. estimate an unrestricted VAR on $(\widehat{\mathbf{f}}_t' \quad \mathbf{z}_t')'$ as explained in Section 3 to get $\widehat{\boldsymbol{\Psi}}(L)$;
3. estimate $(\widehat{\boldsymbol{l}}_i^{f'} \quad \widehat{\boldsymbol{l}}_i^{z'})$ by regressing Δy_{it} onto a constant and the vector $(\Delta\mathbf{f}_t' \quad \Delta\widehat{\mathbf{z}}_t')'$;
4. estimate IRFs as $(\widehat{\boldsymbol{l}}_i^{f'} \quad \widehat{\boldsymbol{l}}_i^{z'})[\widehat{\boldsymbol{\Psi}}(L)]^{-1}$.

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Additional simulation results

Table G1: MonteCarlo Simulations - Impulse Responses
Mean Squared Errors
VECM Estimation – All variables, All Shocks

T	n	δ	$k=0$	$k=1$	$k=4$	$k=8$	$k=12$	$k=16$	$k=20$	$k=100$
75	50	0.50	0.10	0.11	0.21	0.27	0.29	0.30	0.30	0.31
75	50	0.75	0.10	0.12	0.24	0.33	0.39	0.41	0.43	0.45
75	50	0.85	0.11	0.13	0.27	0.40	0.47	0.51	0.53	0.57
75	50	0.95	0.10	0.14	0.29	0.43	0.51	0.55	0.58	0.63
75	50	1.00	0.10	0.15	0.32	0.47	0.57	0.62	0.65	0.70
100	50	0.50	0.07	0.08	0.14	0.19	0.20	0.21	0.21	0.22
100	50	0.75	0.07	0.09	0.17	0.25	0.30	0.33	0.35	0.38
100	50	0.85	0.07	0.09	0.19	0.28	0.34	0.38	0.40	0.46
100	50	0.95	0.07	0.10	0.23	0.35	0.43	0.48	0.51	0.59
100	50	1.00	0.07	0.11	0.25	0.37	0.44	0.50	0.53	0.61
75	75	0.50	0.09	0.11	0.19	0.25	0.26	0.27	0.27	0.27
75	75	0.75	0.09	0.11	0.20	0.29	0.33	0.35	0.36	0.38
75	75	0.85	0.09	0.12	0.24	0.36	0.42	0.46	0.48	0.51
75	75	0.95	0.09	0.13	0.26	0.40	0.48	0.52	0.54	0.58
75	75	1.00	0.08	0.12	0.26	0.41	0.50	0.54	0.57	0.62
100	75	0.50	0.06	0.07	0.14	0.17	0.19	0.19	0.19	0.19
100	75	0.75	0.06	0.07	0.15	0.22	0.25	0.27	0.28	0.30
100	75	0.85	0.06	0.08	0.17	0.26	0.32	0.35	0.37	0.42
100	75	0.95	0.06	0.09	0.19	0.30	0.38	0.42	0.45	0.52
100	75	1.00	0.06	0.09	0.21	0.32	0.40	0.45	0.48	0.56
100	100	0.50	0.06	0.07	0.13	0.16	0.17	0.18	0.18	0.18
100	100	0.75	0.05	0.07	0.14	0.21	0.24	0.26	0.27	0.28
100	100	0.85	0.05	0.07	0.15	0.23	0.28	0.31	0.33	0.37
100	100	0.95	0.06	0.08	0.18	0.29	0.36	0.40	0.43	0.50
100	100	1.00	0.06	0.09	0.19	0.30	0.37	0.42	0.45	0.52
200	200	0.50	0.02	0.03	0.05	0.07	0.07	0.07	0.07	0.07
200	200	0.75	0.02	0.03	0.06	0.09	0.10	0.11	0.12	0.14
200	200	0.85	0.02	0.03	0.07	0.10	0.13	0.15	0.17	0.23
200	200	0.95	0.02	0.04	0.08	0.13	0.16	0.20	0.23	0.35
200	200	1.00	0.02	0.04	0.09	0.14	0.18	0.22	0.26	0.42
300	300	0.50	0.02	0.02	0.03	0.04	0.05	0.05	0.05	0.05
300	300	0.75	0.02	0.02	0.04	0.06	0.07	0.07	0.08	0.10
300	300	0.85	0.02	0.02	0.04	0.06	0.08	0.10	0.11	0.18
300	300	0.95	0.02	0.03	0.05	0.08	0.10	0.12	0.14	0.30
300	300	1.00	0.02	0.03	0.07	0.09	0.11	0.14	0.16	0.34

MSE for the estimated IRFs by fitting a VECM on $\widehat{\mathbf{F}}_t$. T is the number of observations, n is the number of variables, and $n_1 = \lceil n^\delta \rceil$ is the number of $I(1)$ idiosyncratic components. In these simulations there are $n_b = \lceil n^\eta \rceil$ variables with a deterministic linear trend, with $n_b = n_1$.

Table F2a: MonteCarlo Simulations - Impulse Responses
Mean Squared Errors
VECM Estimation – All variables, All Shocks

T	n	δ	η	$k=0$	$k=1$	$k=4$	$k=8$	$k=12$	$k=16$	$k=20$	$k=100$
75	50	0.50	0.50	0.22	0.21	0.35	0.44	0.47	0.48	0.48	0.49
75	50	0.75	0.50	0.51	0.52	0.76	0.84	0.89	0.92	0.95	1.01
75	50	0.75	0.75	0.24	0.22	0.37	0.49	0.54	0.57	0.58	0.59
75	50	0.85	0.50	0.26	0.25	0.41	0.52	0.59	0.62	0.64	0.68
75	50	0.85	0.75	0.30	0.28	0.42	0.52	0.60	0.64	0.66	0.68
75	50	0.85	0.85	0.27	0.26	0.44	0.58	0.65	0.68	0.69	0.73
75	50	0.95	0.50	0.24	0.26	0.43	0.58	0.66	0.71	0.73	0.79
75	50	0.95	0.75	0.32	0.35	0.54	0.70	0.79	0.84	0.87	0.91
75	50	0.95	0.85	0.31	0.30	0.48	0.62	0.69	0.73	0.75	0.80
75	50	0.95	0.95	0.39	0.35	0.56	0.71	0.76	0.79	0.81	0.83
75	50	1.00	0.50	0.25	0.27	0.47	0.61	0.70	0.75	0.78	0.84
75	50	1.00	0.75	0.26	0.30	0.52	0.67	0.76	0.82	0.85	0.91
75	50	1.00	0.85	0.19	0.23	0.41	0.56	0.64	0.68	0.70	0.72
75	50	1.00	0.95	0.23	0.28	0.51	0.67	0.74	0.77	0.78	0.80
75	50	1.00	1.00	0.26	0.26	0.45	0.60	0.68	0.71	0.71	0.74
100	50	0.50	0.50	0.11	0.11	0.20	0.26	0.28	0.29	0.30	0.31
100	50	0.75	0.50	0.17	0.16	0.30	0.38	0.42	0.44	0.46	0.50
100	50	0.75	0.75	0.14	0.14	0.27	0.35	0.40	0.42	0.44	0.47
100	50	0.85	0.50	0.14	0.15	0.28	0.38	0.44	0.48	0.50	0.56
100	50	0.85	0.75	0.11	0.13	0.26	0.37	0.44	0.48	0.51	0.56
100	50	0.85	0.85	0.16	0.16	0.29	0.41	0.47	0.51	0.53	0.57
100	50	0.95	0.50	0.12	0.15	0.30	0.43	0.51	0.56	0.60	0.67
100	50	0.95	0.75	0.14	0.17	0.32	0.44	0.52	0.57	0.61	0.67
100	50	0.95	0.85	0.12	0.15	0.30	0.43	0.51	0.56	0.59	0.65
100	50	0.95	0.95	0.15	0.17	0.31	0.43	0.50	0.54	0.57	0.61
100	50	1.00	0.50	0.14	0.18	0.33	0.46	0.54	0.60	0.64	0.71
100	50	1.00	0.75	0.11	0.14	0.29	0.42	0.50	0.55	0.58	0.64
100	50	1.00	0.85	0.15	0.19	0.35	0.49	0.57	0.61	0.64	0.70
100	50	1.00	0.95	0.15	0.18	0.33	0.46	0.55	0.60	0.63	0.67
100	50	1.00	1.00	0.15	0.18	0.33	0.46	0.54	0.58	0.60	0.64

MSE for the estimated IRFs by fitting a VECM on $\widehat{\mathbf{F}}_t$. T is the number of observations, n is the number of variables, $n_1 = \lceil n^{\delta} \rceil$ is the number of $I(1)$ idiosyncratic components, and $n_b = \lceil n^{\eta} \rceil$ is the number of variables with a deterministic linear trend.

Table F2b: MonteCarlo Simulations - Impulse Responses
Mean Squared Errors
VECM Estimation – All variables, All Shocks

T	n	δ	η	$k=0$	$k=1$	$k=4$	$k=8$	$k=12$	$k=16$	$k=20$	$k=100$
75	75	0.50	0.50	0.19	0.17	0.29	0.35	0.37	0.37	0.37	0.38
75	75	0.75	0.50	0.21	0.22	0.38	0.45	0.47	0.49	0.50	0.52
75	75	0.75	0.75	0.20	0.21	0.36	0.46	0.50	0.52	0.53	0.55
75	75	0.85	0.50	0.28	0.26	0.41	0.50	0.54	0.58	0.60	0.64
75	75	0.85	0.75	0.23	0.22	0.38	0.48	0.53	0.56	0.57	0.60
75	75	0.85	0.85	0.21	0.19	0.35	0.47	0.54	0.57	0.58	0.60
75	75	0.95	0.50	0.22	0.23	0.38	0.51	0.59	0.64	0.66	0.71
75	75	0.95	0.75	0.25	0.26	0.42	0.53	0.60	0.64	0.67	0.70
75	75	0.95	0.85	0.16	0.17	0.34	0.49	0.57	0.62	0.63	0.66
75	75	0.95	0.95	0.18	0.19	0.36	0.51	0.59	0.62	0.63	0.64
75	75	1.00	0.50	0.17	0.21	0.38	0.52	0.61	0.66	0.68	0.73
75	75	1.00	0.75	0.19	0.21	0.38	0.51	0.60	0.64	0.66	0.69
75	75	1.00	0.85	0.17	0.20	0.38	0.54	0.63	0.67	0.69	0.72
75	75	1.00	0.95	0.13	0.16	0.33	0.49	0.57	0.60	0.61	0.63
75	75	1.00	1.00	0.18	0.22	0.37	0.52	0.59	0.61	0.63	0.64
100	75	0.50	0.50	0.09	0.10	0.18	0.22	0.23	0.24	0.24	0.24
100	75	0.75	0.50	0.10	0.11	0.21	0.28	0.32	0.34	0.35	0.37
100	75	0.75	0.75	0.11	0.12	0.22	0.29	0.32	0.34	0.35	0.37
100	75	0.85	0.50	0.10	0.11	0.21	0.30	0.36	0.39	0.41	0.46
100	75	0.85	0.75	0.15	0.14	0.25	0.34	0.39	0.42	0.44	0.47
100	75	0.85	0.85	0.11	0.12	0.22	0.32	0.37	0.41	0.42	0.45
100	75	0.95	0.50	0.09	0.12	0.25	0.36	0.42	0.47	0.50	0.56
100	75	0.95	0.75	0.19	0.21	0.34	0.42	0.48	0.51	0.54	0.58
100	75	0.95	0.85	0.12	0.14	0.26	0.39	0.46	0.51	0.54	0.60
100	75	0.95	0.95	0.09	0.11	0.24	0.36	0.43	0.47	0.50	0.53
100	75	1.00	0.50	0.11	0.14	0.27	0.39	0.47	0.52	0.55	0.62
100	75	1.00	0.75	0.13	0.16	0.31	0.41	0.49	0.54	0.57	0.64
100	75	1.00	0.85	0.11	0.14	0.27	0.40	0.48	0.53	0.57	0.62
100	75	1.00	0.95	0.11	0.13	0.26	0.39	0.46	0.51	0.53	0.57
100	75	1.00	1.00	0.09	0.12	0.25	0.38	0.46	0.51	0.53	0.57

MSE for the estimated IRFs by fitting a VECM on $\widehat{\mathbf{F}}_t$. T is the number of observations, n is the number of variables, $n_1 = \lceil n^{\delta} \rceil$ is the number of $I(1)$ idiosyncratic components, and $n_b = \lceil n^{\eta} \rceil$ is the number of variables with a deterministic linear trend.

Table F2c: MonteCarlo Simulations - Impulse Responses
Mean Squared Errors
VECM Estimation – All variables, All Shocks

T	n	δ	η	$k=0$	$k=1$	$k=4$	$k=8$	$k=12$	$k=16$	$k=20$	$k=100$
100	100	0.50	0.50	0.09	0.10	0.17	0.21	0.22	0.22	0.22	0.23
100	100	0.75	0.50	0.09	0.10	0.18	0.25	0.28	0.30	0.31	0.33
100	100	0.75	0.75	0.09	0.10	0.20	0.27	0.30	0.32	0.33	0.34
100	100	0.85	0.50	0.11	0.13	0.24	0.32	0.37	0.40	0.42	0.46
100	100	0.85	0.75	0.10	0.11	0.22	0.31	0.36	0.39	0.41	0.45
100	100	0.85	0.85	0.10	0.11	0.22	0.31	0.36	0.39	0.41	0.44
100	100	0.95	0.50	0.09	0.11	0.21	0.33	0.40	0.44	0.48	0.54
100	100	0.95	0.75	0.09	0.10	0.21	0.31	0.39	0.43	0.46	0.51
100	100	0.95	0.85	0.09	0.12	0.23	0.34	0.42	0.47	0.50	0.55
100	100	0.95	0.95	0.09	0.11	0.22	0.33	0.41	0.45	0.47	0.51
100	100	1.00	0.50	0.08	0.10	0.22	0.33	0.40	0.45	0.48	0.54
100	100	1.00	0.75	0.09	0.11	0.23	0.36	0.44	0.50	0.53	0.59
100	100	1.00	0.85	0.08	0.10	0.22	0.34	0.41	0.46	0.49	0.55
100	100	1.00	0.95	0.08	0.10	0.23	0.37	0.45	0.50	0.52	0.56
100	100	1.00	1.00	0.08	0.11	0.22	0.34	0.41	0.45	0.46	0.49
200	200	0.50	0.50	0.04	0.04	0.07	0.09	0.09	0.10	0.10	0.10
200	200	0.75	0.50	0.03	0.04	0.07	0.10	0.12	0.13	0.14	0.16
200	200	0.75	0.75	0.03	0.04	0.07	0.10	0.11	0.12	0.13	0.15
200	200	0.85	0.50	0.03	0.04	0.08	0.12	0.15	0.17	0.18	0.25
200	200	0.85	0.75	0.03	0.04	0.08	0.12	0.15	0.17	0.19	0.25
200	200	0.85	0.85	0.03	0.04	0.08	0.11	0.14	0.17	0.18	0.24
200	200	0.95	0.50	0.03	0.04	0.09	0.14	0.17	0.21	0.23	0.36
200	200	0.95	0.75	0.03	0.05	0.09	0.14	0.18	0.22	0.25	0.38
200	200	0.95	0.85	0.03	0.04	0.09	0.14	0.19	0.22	0.26	0.38
200	200	0.95	0.95	0.03	0.04	0.09	0.15	0.19	0.23	0.26	0.38
200	200	1.00	0.50	0.03	0.05	0.10	0.15	0.19	0.23	0.26	0.41
200	200	1.00	0.75	0.03	0.04	0.10	0.15	0.19	0.23	0.26	0.41
200	200	1.00	0.85	0.03	0.05	0.11	0.16	0.21	0.26	0.30	0.45
200	200	1.00	0.95	0.03	0.05	0.10	0.16	0.21	0.26	0.29	0.43
200	200	1.00	1.00	0.03	0.04	0.10	0.16	0.21	0.25	0.28	0.40
300	300	0.50	0.50	0.02	0.02	0.04	0.05	0.06	0.06	0.06	0.06
300	300	0.75	0.50	0.02	0.02	0.04	0.06	0.07	0.08	0.08	0.10
300	300	0.75	0.75	0.02	0.02	0.05	0.06	0.07	0.08	0.08	0.11
300	300	0.85	0.50	0.02	0.03	0.05	0.07	0.09	0.10	0.12	0.19
300	300	0.85	0.75	0.02	0.03	0.05	0.07	0.09	0.11	0.12	0.19
300	300	0.85	0.85	0.02	0.03	0.05	0.07	0.09	0.10	0.12	0.18
300	300	0.95	0.50	0.02	0.03	0.06	0.09	0.11	0.13	0.16	0.33
300	300	0.95	0.75	0.02	0.03	0.06	0.09	0.11	0.14	0.16	0.32
300	300	0.95	0.85	0.02	0.03	0.06	0.09	0.12	0.14	0.17	0.33
300	300	0.95	0.95	0.02	0.03	0.06	0.09	0.12	0.15	0.18	0.33
300	300	1.00	0.50	0.02	0.03	0.07	0.09	0.12	0.14	0.17	0.36
300	300	1.00	0.75	0.02	0.03	0.07	0.10	0.12	0.15	0.17	0.36
300	300	1.00	0.85	0.02	0.03	0.07	0.10	0.12	0.15	0.18	0.36
300	300	1.00	0.95	0.02	0.03	0.07	0.10	0.13	0.16	0.19	0.37
300	300	1.00	1.00	0.02	0.03	0.06	0.10	0.13	0.16	0.19	0.36

MSE for the estimated IRFs by fitting a VECM on $\widehat{\mathbf{F}}_t$. T is the number of observations, n is the number of variables, $n_1 = \lceil n^\delta \rceil$ is the number of $I(1)$ idiosyncratic components, and $n_b = \lceil n^\eta \rceil$ is the number of variables with a deterministic linear trend.

Table F3a: MonteCarlo Simulations - Impulse Responses
Mean Squared Errors
VECM Estimation – First variable, first shock

T	n	δ	η	$k=0$	$k=1$	$k=4$	$k=8$	$k=12$	$k=16$	$k=20$	$k=100$
75	50	0.50	0.50	0.00	0.03	0.08	0.10	0.11	0.11	0.11	0.11
75	50	0.75	0.50	0.00	0.03	0.09	0.14	0.16	0.17	0.18	0.19
75	50	0.75	0.75	0.00	0.03	0.09	0.14	0.16	0.17	0.18	0.18
75	50	0.85	0.50	0.00	0.03	0.10	0.16	0.19	0.20	0.21	0.22
75	50	0.85	0.75	0.00	0.03	0.11	0.18	0.21	0.23	0.23	0.24
75	50	0.85	0.85	0.00	0.03	0.12	0.19	0.22	0.23	0.24	0.25
75	50	0.95	0.50	0.00	0.03	0.11	0.19	0.23	0.25	0.26	0.27
75	50	0.95	0.75	0.00	0.03	0.11	0.19	0.24	0.26	0.26	0.27
75	50	0.95	0.85	0.00	0.03	0.12	0.21	0.24	0.26	0.26	0.27
75	50	0.95	0.95	0.00	0.03	0.12	0.20	0.23	0.24	0.24	0.25
75	50	1.00	0.50	0.00	0.03	0.12	0.21	0.26	0.28	0.29	0.30
75	50	1.00	0.75	0.00	0.03	0.12	0.20	0.25	0.27	0.27	0.28
75	50	1.00	0.85	0.00	0.04	0.12	0.21	0.26	0.28	0.28	0.29
75	50	1.00	0.95	0.00	0.03	0.12	0.20	0.24	0.25	0.26	0.26
75	50	1.00	1.00	0.00	0.03	0.12	0.20	0.24	0.25	0.25	0.26
100	50	0.50	0.50	0.00	0.02	0.06	0.09	0.10	0.10	0.10	0.10
100	50	0.75	0.50	0.00	0.03	0.07	0.11	0.13	0.15	0.16	0.17
100	50	0.75	0.75	0.00	0.03	0.08	0.12	0.15	0.16	0.17	0.18
100	50	0.85	0.50	0.00	0.02	0.07	0.13	0.16	0.18	0.19	0.22
100	50	0.85	0.75	0.00	0.03	0.08	0.14	0.18	0.20	0.21	0.24
100	50	0.85	0.85	0.00	0.03	0.09	0.16	0.20	0.22	0.23	0.25
100	50	0.95	0.50	0.00	0.03	0.09	0.15	0.20	0.23	0.24	0.27
100	50	0.95	0.75	0.00	0.03	0.10	0.17	0.22	0.24	0.26	0.28
100	50	0.95	0.85	0.00	0.03	0.09	0.16	0.21	0.24	0.25	0.27
100	50	0.95	0.95	0.00	0.03	0.09	0.15	0.19	0.21	0.22	0.24
100	50	1.00	0.50	0.00	0.03	0.09	0.16	0.21	0.24	0.25	0.29
100	50	1.00	0.75	0.00	0.03	0.09	0.16	0.20	0.23	0.24	0.26
100	50	1.00	0.85	0.00	0.03	0.09	0.16	0.21	0.23	0.25	0.27
100	50	1.00	0.95	0.00	0.03	0.10	0.18	0.22	0.25	0.26	0.28
100	50	1.00	1.00	0.00	0.03	0.10	0.18	0.22	0.24	0.25	0.27

MSE for the estimated IRF $\widehat{\phi}_{11}(L)$ by fitting a VECM on $\widehat{\mathbf{F}}_t$. T is the number of observations, n is the number of variables, $n_1 = \lceil n^{\delta} \rceil$ is the number of $I(1)$ idiosyncratic components, and $n_b = \lceil n^{\eta} \rceil$ is the number of variables with a deterministic linear trend.

Table F3b: MonteCarlo Simulations - Impulse Responses
Mean Squared Errors
VECM Estimation – First variable, first shock

T	n	δ	η	$k=0$	$k=1$	$k=4$	$k=8$	$k=12$	$k=16$	$k=20$	$k=100$
75	75	0.50	0.50	0.00	0.03	0.07	0.09	0.09	0.10	0.10	0.10
75	75	0.75	0.50	0.00	0.03	0.08	0.12	0.14	0.14	0.15	0.15
75	75	0.75	0.75	0.00	0.03	0.09	0.13	0.14	0.15	0.15	0.16
75	75	0.85	0.50	0.00	0.03	0.09	0.16	0.19	0.20	0.21	0.22
75	75	0.85	0.75	0.00	0.03	0.10	0.17	0.20	0.21	0.22	0.23
75	75	0.85	0.85	0.00	0.03	0.10	0.17	0.20	0.21	0.21	0.22
75	75	0.95	0.50	0.00	0.03	0.11	0.18	0.22	0.24	0.25	0.26
75	75	0.95	0.75	0.00	0.03	0.11	0.19	0.24	0.26	0.26	0.27
75	75	0.95	0.85	0.00	0.03	0.10	0.17	0.21	0.22	0.23	0.23
75	75	0.95	0.95	0.00	0.03	0.12	0.21	0.25	0.26	0.26	0.27
75	75	1.00	0.50	0.00	0.03	0.10	0.18	0.23	0.25	0.26	0.27
75	75	1.00	0.75	0.00	0.03	0.11	0.20	0.24	0.26	0.27	0.28
75	75	1.00	0.85	0.00	0.03	0.11	0.21	0.25	0.27	0.27	0.28
75	75	1.00	0.95	0.00	0.03	0.12	0.21	0.25	0.26	0.27	0.27
75	75	1.00	1.00	0.00	0.03	0.12	0.21	0.25	0.26	0.26	0.27
100	75	0.50	0.50	0.00	0.02	0.05	0.06	0.07	0.07	0.07	0.07
100	75	0.75	0.50	0.00	0.02	0.06	0.10	0.12	0.13	0.13	0.14
100	75	0.75	0.75	0.00	0.02	0.07	0.10	0.12	0.13	0.14	0.15
100	75	0.85	0.50	0.00	0.02	0.07	0.12	0.15	0.17	0.19	0.21
100	75	0.85	0.75	0.00	0.02	0.07	0.13	0.16	0.18	0.19	0.20
100	75	0.85	0.85	0.00	0.02	0.07	0.13	0.16	0.18	0.19	0.21
100	75	0.95	0.50	0.00	0.02	0.08	0.14	0.18	0.21	0.23	0.26
100	75	0.95	0.75	0.00	0.02	0.08	0.15	0.19	0.22	0.23	0.25
100	75	0.95	0.85	0.00	0.02	0.08	0.15	0.19	0.21	0.22	0.24
100	75	0.95	0.95	0.00	0.03	0.09	0.16	0.21	0.23	0.24	0.26
100	75	1.00	0.50	0.00	0.03	0.08	0.14	0.18	0.21	0.22	0.26
100	75	1.00	0.75	0.00	0.02	0.08	0.15	0.20	0.22	0.23	0.26
100	75	1.00	0.85	0.00	0.02	0.08	0.15	0.19	0.22	0.23	0.25
100	75	1.00	0.95	0.00	0.02	0.08	0.16	0.21	0.24	0.25	0.26
100	75	1.00	1.00	0.00	0.02	0.08	0.15	0.19	0.21	0.22	0.24

MSE for the estimated IRF $\widehat{\phi}_{11}(L)$ by fitting a VECM on $\widehat{\mathbf{F}}_t$. T is the number of observations, n is the number of variables, $n_1 = \lceil n^{\delta} \rceil$ is the number of $I(1)$ idiosyncratic components, and $n_b = \lceil n^{\eta} \rceil$ is the number of variables with a deterministic linear trend.

Table F3c: MonteCarlo Simulations - Impulse Responses
Mean Squared Errors
VECM Estimation – First variable, first shock

T	n	δ	η	$k=0$	$k=1$	$k=4$	$k=8$	$k=12$	$k=16$	$k=20$	$k=100$
100	100	0.50	0.50	0.00	0.02	0.05	0.06	0.06	0.07	0.07	0.07
100	100	0.75	0.50	0.00	0.02	0.06	0.09	0.10	0.11	0.11	0.12
100	100	0.75	0.75	0.00	0.02	0.06	0.09	0.10	0.11	0.11	0.12
100	100	0.85	0.50	0.00	0.02	0.06	0.11	0.14	0.15	0.16	0.18
100	100	0.85	0.75	0.00	0.02	0.07	0.11	0.14	0.16	0.17	0.18
100	100	0.85	0.85	0.00	0.02	0.07	0.12	0.15	0.17	0.18	0.19
100	100	0.95	0.50	0.00	0.02	0.07	0.12	0.17	0.19	0.21	0.24
100	100	0.95	0.75	0.00	0.02	0.07	0.13	0.18	0.20	0.21	0.24
100	100	0.95	0.85	0.00	0.02	0.08	0.14	0.19	0.21	0.23	0.25
100	100	0.95	0.95	0.00	0.02	0.08	0.15	0.20	0.22	0.23	0.25
100	100	1.00	0.50	0.00	0.02	0.07	0.14	0.18	0.21	0.22	0.25
100	100	1.00	0.75	0.00	0.02	0.08	0.16	0.21	0.23	0.25	0.27
100	100	1.00	0.85	0.00	0.02	0.08	0.16	0.21	0.23	0.25	0.27
100	100	1.00	0.95	0.00	0.02	0.08	0.16	0.21	0.23	0.24	0.26
100	100	1.00	1.00	0.00	0.02	0.09	0.17	0.22	0.24	0.25	0.27
200	200	0.50	0.50	0.00	0.01	0.02	0.03	0.03	0.03	0.03	0.03
200	200	0.75	0.50	0.00	0.01	0.02	0.04	0.04	0.05	0.05	0.06
200	200	0.75	0.75	0.00	0.01	0.02	0.04	0.04	0.05	0.05	0.06
200	200	0.85	0.50	0.00	0.01	0.03	0.05	0.06	0.08	0.09	0.13
200	200	0.85	0.75	0.00	0.01	0.03	0.05	0.07	0.08	0.09	0.13
200	200	0.85	0.85	0.00	0.01	0.03	0.05	0.06	0.08	0.09	0.12
200	200	0.95	0.50	0.00	0.01	0.03	0.06	0.08	0.10	0.12	0.20
200	200	0.95	0.75	0.00	0.01	0.03	0.06	0.08	0.11	0.12	0.19
200	200	0.95	0.85	0.00	0.01	0.03	0.06	0.09	0.12	0.14	0.22
200	200	0.95	0.95	0.00	0.01	0.03	0.06	0.09	0.12	0.14	0.21
200	200	1.00	0.50	0.00	0.01	0.03	0.06	0.09	0.12	0.14	0.22
200	200	1.00	0.75	0.00	0.01	0.04	0.06	0.09	0.11	0.13	0.22
200	200	1.00	0.85	0.00	0.01	0.03	0.06	0.09	0.12	0.14	0.22
200	200	1.00	0.95	0.00	0.01	0.04	0.07	0.11	0.14	0.16	0.23
200	200	1.00	1.00	0.00	0.01	0.03	0.07	0.10	0.13	0.15	0.22
300	300	0.50	0.50	0.00	0.01	0.01	0.02	0.02	0.02	0.02	0.02
300	300	0.75	0.50	0.00	0.01	0.01	0.02	0.02	0.03	0.03	0.04
300	300	0.75	0.75	0.00	0.01	0.02	0.02	0.03	0.03	0.03	0.04
300	300	0.85	0.50	0.00	0.01	0.02	0.03	0.04	0.05	0.05	0.09
300	300	0.85	0.75	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.10
300	300	0.85	0.85	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.10
300	300	0.95	0.50	0.00	0.01	0.02	0.04	0.05	0.07	0.08	0.18
300	300	0.95	0.75	0.00	0.01	0.02	0.04	0.06	0.07	0.09	0.18
300	300	0.95	0.85	0.00	0.01	0.02	0.04	0.05	0.07	0.09	0.18
300	300	0.95	0.95	0.00	0.01	0.02	0.04	0.06	0.08	0.09	0.18
300	300	1.00	0.50	0.00	0.01	0.02	0.03	0.05	0.07	0.08	0.19
300	300	1.00	0.75	0.00	0.01	0.02	0.04	0.06	0.08	0.09	0.21
300	300	1.00	0.85	0.00	0.01	0.02	0.04	0.06	0.08	0.10	0.20
300	300	1.00	0.95	0.00	0.01	0.02	0.04	0.07	0.09	0.11	0.22
300	300	1.00	1.00	0.00	0.01	0.02	0.04	0.06	0.08	0.10	0.19

MSE for the estimated IRF $\widehat{\phi}_{11}(L)$ by fitting a VECM on $\widehat{\mathbf{F}}_t$. T is the number of observations, n is the number of variables, $n_1 = \lceil n^{\delta} \rceil$ is the number of $I(1)$ idiosyncratic components, and $n_b = \lceil n^{\eta} \rceil$ is the number of variables with a deterministic linear trend.

Table F4: MonteCarlo Simulations - Impulse Responses
Mean Squared Errors relative to Bai and Ng (2004)
VECM Estimation – All variables, All Shocks

T	n	δ	$k=0$	$k=1$	$k=4$	$k=8$	$k=12$	$k=16$	$k=20$	$k=100$
75	50	0.50	1.01	0.96	0.88	0.82	0.79	0.77	0.76	0.74
75	50	0.75	1.01	0.96	0.91	0.89	0.89	0.88	0.87	0.85
75	50	0.85	0.98	0.93	0.91	0.95	0.95	0.95	0.94	0.94
75	50	0.95	1.01	0.95	0.94	0.96	0.96	0.96	0.96	0.96
75	50	1.00	1.01	0.97	0.96	0.99	1.01	1.01	1.00	0.99
100	50	0.50	0.99	0.92	0.80	0.77	0.74	0.72	0.71	0.68
100	50	0.75	1.00	0.94	0.87	0.89	0.89	0.88	0.87	0.85
100	50	0.85	1.00	0.95	0.89	0.90	0.91	0.92	0.91	0.88
100	50	0.95	0.99	0.95	0.92	0.96	0.98	0.98	0.97	0.95
100	50	1.00	0.99	0.95	0.92	0.96	0.98	0.97	0.97	0.94
75	75	0.50	0.98	0.94	0.82	0.75	0.72	0.70	0.69	0.68
75	75	0.75	1.01	0.97	0.86	0.84	0.82	0.81	0.80	0.78
75	75	0.85	0.99	0.95	0.90	0.93	0.93	0.93	0.93	0.91
75	75	0.95	0.96	0.94	0.92	0.93	0.93	0.93	0.92	0.90
75	75	1.00	0.99	0.97	0.94	0.96	0.97	0.97	0.96	0.95
100	75	0.50	0.99	0.92	0.82	0.76	0.72	0.70	0.69	0.67
100	75	0.75	1.00	0.95	0.84	0.83	0.82	0.80	0.79	0.76
100	75	0.85	1.00	0.93	0.86	0.89	0.90	0.90	0.89	0.86
100	75	0.95	1.00	0.95	0.89	0.93	0.95	0.96	0.95	0.92
100	75	1.00	1.00	0.95	0.92	0.96	0.97	0.97	0.96	0.94
100	100	0.50	0.99	0.94	0.80	0.74	0.70	0.68	0.66	0.64
100	100	0.75	0.99	0.93	0.85	0.83	0.82	0.80	0.79	0.76
100	100	0.85	0.99	0.93	0.87	0.88	0.88	0.87	0.86	0.82
100	100	0.95	1.00	0.96	0.91	0.95	0.95	0.95	0.94	0.90
100	100	1.00	0.99	0.95	0.90	0.92	0.93	0.93	0.93	0.92
200	200	0.50	0.98	0.88	0.72	0.69	0.66	0.63	0.61	0.57
200	200	0.75	0.99	0.87	0.73	0.75	0.74	0.72	0.70	0.62
200	200	0.85	0.98	0.87	0.77	0.81	0.81	0.81	0.81	0.74
200	200	0.95	0.99	0.89	0.81	0.88	0.91	0.93	0.94	0.88
200	200	1.00	0.99	0.90	0.82	0.89	0.93	0.95	0.96	0.94
300	300	0.50	0.98	0.85	0.69	0.70	0.67	0.64	0.62	0.55
300	300	0.75	0.98	0.84	0.71	0.75	0.74	0.73	0.71	0.61
300	300	0.85	0.98	0.83	0.71	0.78	0.80	0.81	0.81	0.75
300	300	0.95	0.98	0.85	0.73	0.82	0.86	0.88	0.90	0.88
300	300	1.00	0.98	0.87	0.78	0.85	0.90	0.93	0.95	0.91

Ratio between the MSE for the estimated IRFs obtained by fitting a VECM on $\widehat{\mathbf{F}}_t$, and the MSE for the estimated IRFs obtained by fitting a VECM on the common factors estimated as in Bai and Ng (2004). Values smaller than one indicate a better performance of our method. T is the number of observations, n is the number of variables, and $n_1 = \lceil n^\delta \rceil$ is the number of $I(1)$ idiosyncratic components. In these simulations there are $n_b = \lceil n^\eta \rceil$ variables with a deterministic linear trend, with $n_b = n_1$.

Table F5a: MonteCarlo Simulations - Impulse Responses
Mean Squared Errors relative to Bai and Ng (2004)
VECM Estimation – All variables, All Shocks

T	n	δ	η	$k=0$	$k=1$	$k=4$	$k=8$	$k=12$	$k=16$	$k=20$	$k=100$
75	50	0.50	0.50	0.98	0.94	0.87	0.85	0.84	0.84	0.84	0.84
75	50	0.75	0.50	1.44	1.40	1.42	1.37	1.32	1.31	1.31	1.31
75	50	0.75	0.75	1.15	1.06	1.04	1.03	1.03	1.02	1.00	0.98
75	50	0.85	0.50	1.17	1.08	1.05	1.03	1.02	1.01	1.01	1.00
75	50	0.85	0.75	0.92	0.85	0.84	0.85	0.88	0.88	0.86	0.79
75	50	0.85	0.85	1.16	1.09	1.08	1.09	1.08	1.06	1.04	1.03
75	50	0.95	0.50	1.06	1.03	1.02	1.06	1.06	1.05	1.05	1.04
75	50	0.95	0.75	0.70	0.71	0.78	0.86	0.91	0.93	0.92	0.94
75	50	0.95	0.85	1.06	0.97	1.00	1.07	1.05	1.04	1.03	0.98
75	50	0.95	0.95	1.05	0.96	0.97	0.98	0.98	0.98	0.97	0.93
75	50	1.00	0.50	1.01	0.97	1.02	1.04	1.03	1.03	1.02	1.00
75	50	1.00	0.75	1.10	1.07	1.09	1.10	1.11	1.11	1.10	1.07
75	50	1.00	0.85	1.03	1.01	1.01	1.03	1.02	1.00	0.98	0.94
75	50	1.00	0.95	0.97	0.96	1.01	1.05	1.06	1.05	1.02	0.98
75	50	1.00	1.00	1.07	1.00	1.03	1.06	1.03	1.00	0.96	0.90
100	50	0.50	0.50	0.97	0.94	0.87	0.85	0.84	0.83	0.83	0.81
100	50	0.75	0.50	1.09	1.01	1.01	1.00	0.98	0.97	0.96	0.92
100	50	0.75	0.75	0.93	0.91	0.89	0.90	0.90	0.90	0.89	0.86
100	50	0.85	0.50	1.01	0.96	0.95	0.97	0.98	0.97	0.96	0.93
100	50	0.85	0.75	0.97	0.93	0.91	0.94	0.96	0.96	0.95	0.91
100	50	0.85	0.85	1.04	0.96	0.89	0.93	0.95	0.96	0.96	0.91
100	50	0.95	0.50	1.00	0.97	0.94	0.97	1.00	1.00	1.00	0.96
100	50	0.95	0.75	1.01	0.98	0.94	0.98	0.99	0.99	0.99	0.93
100	50	0.95	0.85	1.04	1.00	0.99	1.03	1.04	1.05	1.04	0.96
100	50	0.95	0.95	0.85	0.84	0.89	0.94	0.96	0.96	0.94	0.88
100	50	1.00	0.50	1.04	1.02	0.99	1.00	1.01	1.01	1.01	0.97
100	50	1.00	0.75	0.97	0.96	0.94	1.00	1.01	1.01	0.99	0.92
100	50	1.00	0.85	1.04	1.00	0.98	1.03	1.04	1.04	1.03	0.99
100	50	1.00	0.95	1.07	1.02	0.96	1.01	1.05	1.04	1.02	0.93
100	50	1.00	1.00	1.08	0.99	0.99	1.03	1.03	1.01	0.99	0.90

Ratio between the MSE for the estimated IRFs obtained by fitting a VECM on $\widehat{\mathbf{F}}_t$, and the MSE for the estimated IRFs obtained by fitting a VECM on the common factors estimated as in Bai and Ng (2004). Values smaller than one indicate a better performance of our method. T is the number of observations, n is the number of variables, and $n_1 = \lceil n^\delta \rceil$ is the number of $I(1)$ idiosyncratic components, and $n_b = \lceil n^\eta \rceil$ is the number of variables with a deterministic linear trend.

Table F5b: MonteCarlo Simulations - Impulse Responses
Mean Squared Errors relative to Bai and Ng (2004)
VECM Estimation – All variables, All Shocks

T	n	δ	η	$k=0$	$k=1$	$k=4$	$k=8$	$k=12$	$k=16$	$k=20$	$k=100$
75	75	0.50	0.50	0.96	0.91	0.89	0.84	0.81	0.79	0.78	0.78
75	75	0.75	0.50	0.98	0.97	1.02	0.99	0.95	0.94	0.93	0.92
75	75	0.75	0.75	1.18	1.18	1.08	1.05	1.03	1.00	0.99	0.96
75	75	0.85	0.50	0.89	0.94	0.91	0.91	0.88	0.87	0.86	0.86
75	75	0.85	0.75	1.01	0.99	0.97	0.93	0.93	0.92	0.92	0.90
75	75	0.85	0.85	1.12	1.03	1.00	1.01	1.01	1.00	0.98	0.94
75	75	0.95	0.50	0.94	0.95	0.95	0.96	0.97	0.97	0.96	0.94
75	75	0.95	0.75	1.10	1.06	0.97	0.97	0.97	0.96	0.94	0.92
75	75	0.95	0.85	1.03	0.95	0.95	0.99	1.01	1.00	0.98	0.94
75	75	0.95	0.95	0.98	0.95	0.98	1.03	1.03	1.00	0.97	0.90
75	75	1.00	0.50	0.97	0.96	0.96	0.98	0.98	0.98	0.97	0.94
75	75	1.00	0.75	0.99	0.93	0.92	0.98	1.02	1.02	1.00	0.96
75	75	1.00	0.85	1.04	0.99	0.99	1.01	1.02	1.02	1.00	0.96
75	75	1.00	0.95	0.96	0.90	0.94	0.98	0.97	0.94	0.91	0.84
75	75	1.00	1.00	1.09	1.04	0.95	1.01	1.00	0.97	0.94	0.89
100	75	0.50	0.50	0.97	0.91	0.86	0.82	0.79	0.77	0.76	0.75
100	75	0.75	0.50	1.02	0.97	0.93	0.91	0.88	0.88	0.87	0.84
100	75	0.75	0.75	0.99	0.99	0.95	0.93	0.92	0.89	0.88	0.83
100	75	0.85	0.50	0.94	0.91	0.87	0.89	0.90	0.89	0.88	0.85
100	75	0.85	0.75	1.08	0.97	0.93	0.95	0.96	0.95	0.93	0.88
100	75	0.85	0.85	1.03	0.97	0.90	0.93	0.94	0.93	0.91	0.85
100	75	0.95	0.50	1.00	0.97	0.93	0.95	0.95	0.95	0.95	0.91
100	75	0.95	0.75	1.76	1.52	1.25	1.14	1.08	1.04	1.01	0.95
100	75	0.95	0.85	1.03	0.97	0.97	1.02	1.03	1.01	1.00	0.92
100	75	0.95	0.95	1.01	0.94	0.93	1.01	1.03	1.02	1.00	0.92
100	75	1.00	0.50	1.10	1.04	0.96	0.99	1.00	1.00	1.00	0.95
100	75	1.00	0.75	1.09	0.98	0.97	0.99	1.01	1.01	1.00	0.94
100	75	1.00	0.85	0.98	0.94	0.94	0.99	1.01	1.01	1.00	0.93
100	75	1.00	0.95	1.02	0.95	0.93	1.00	1.02	1.00	0.98	0.91
100	75	1.00	1.00	1.02	0.95	0.96	1.03	1.03	1.01	0.99	0.89

Ratio between the MSE for the estimated IRFs obtained by fitting a VECM on $\widehat{\mathbf{F}}_t$, and the MSE for the estimated IRFs obtained by fitting a VECM on the common factors estimated as in Bai and Ng (2004). Values smaller than one indicate a better performance of our method. T is the number of observations, n is the number of variables, and $n_1 = \lceil n^{\delta} \rceil$ is the number of $I(1)$ idiosyncratic components, and $n_b = \lceil n^{\eta} \rceil$ is the number of variables with a deterministic linear trend.

Table F5c: MonteCarlo Simulations - Impulse Responses
Mean Squared Errors relative to Bai and Ng (2004)
VECM Estimation – All variables, All Shocks

T	n	δ	η	$k=0$	$k=1$	$k=4$	$k=8$	$k=12$	$k=16$	$k=20$	$k=100$
100	100	0.50	0.50	1.00	0.95	0.85	0.79	0.75	0.73	0.72	0.70
100	100	0.75	0.50	1.01	0.95	0.88	0.86	0.85	0.84	0.83	0.81
100	100	0.75	0.75	0.96	0.94	0.89	0.85	0.83	0.82	0.81	0.78
100	100	0.85	0.50	1.01	0.96	0.93	0.92	0.92	0.92	0.92	0.89
100	100	0.85	0.75	0.97	0.92	0.93	0.95	0.95	0.94	0.93	0.87
100	100	0.85	0.85	1.04	0.98	0.94	0.96	0.95	0.93	0.92	0.86
100	100	0.95	0.50	0.99	0.93	0.90	0.92	0.93	0.93	0.92	0.89
100	100	0.95	0.75	1.03	0.95	0.91	0.94	0.95	0.95	0.94	0.88
100	100	0.95	0.85	1.00	0.96	0.95	1.01	1.03	1.03	1.01	0.93
100	100	0.95	0.95	1.03	0.95	0.93	0.99	1.00	0.99	0.96	0.88
100	100	1.00	0.50	1.00	0.95	0.92	0.96	0.97	0.96	0.95	0.91
100	100	1.00	0.75	1.00	0.94	0.92	0.97	0.99	0.99	0.98	0.93
100	100	1.00	0.85	1.00	0.94	0.93	0.98	1.00	0.98	0.97	0.89
100	100	1.00	0.95	0.99	0.93	0.95	1.02	1.04	1.02	0.99	0.89
100	100	1.00	1.00	1.00	0.94	0.93	0.99	0.99	0.97	0.94	0.85
200	200	0.50	0.50	0.98	0.91	0.80	0.78	0.74	0.71	0.70	0.66
200	200	0.75	0.50	0.99	0.90	0.79	0.80	0.79	0.78	0.77	0.72
200	200	0.75	0.75	0.99	0.90	0.77	0.77	0.75	0.73	0.71	0.65
200	200	0.85	0.50	0.99	0.90	0.81	0.85	0.86	0.86	0.86	0.78
200	200	0.85	0.75	0.99	0.89	0.81	0.86	0.88	0.88	0.88	0.79
200	200	0.85	0.85	0.99	0.88	0.80	0.86	0.88	0.88	0.88	0.79
200	200	0.95	0.50	0.99	0.90	0.81	0.87	0.90	0.92	0.93	0.88
200	200	0.95	0.75	0.99	0.90	0.82	0.89	0.94	0.97	0.98	0.91
200	200	0.95	0.85	0.99	0.90	0.83	0.93	0.98	1.01	1.01	0.92
200	200	0.95	0.95	0.99	0.89	0.82	0.93	0.99	1.01	1.02	0.89
200	200	1.00	0.50	0.99	0.91	0.84	0.91	0.95	0.97	0.98	0.93
200	200	1.00	0.75	0.99	0.89	0.82	0.90	0.95	0.97	0.98	0.92
200	200	1.00	0.85	0.99	0.91	0.84	0.94	1.00	1.03	1.04	0.93
200	200	1.00	0.95	0.99	0.89	0.84	0.96	1.02	1.05	1.06	0.92
200	200	1.00	1.00	0.98	0.87	0.82	0.95	1.02	1.05	1.05	0.90
300	300	0.50	0.50	0.98	0.87	0.75	0.75	0.72	0.70	0.68	0.63
300	300	0.75	0.50	0.99	0.85	0.74	0.78	0.77	0.75	0.73	0.64
300	300	0.75	0.75	0.98	0.85	0.73	0.76	0.76	0.75	0.73	0.65
300	300	0.85	0.50	0.99	0.86	0.75	0.80	0.82	0.83	0.83	0.77
300	300	0.85	0.75	0.99	0.86	0.76	0.82	0.84	0.85	0.85	0.78
300	300	0.85	0.85	0.98	0.85	0.75	0.80	0.83	0.84	0.85	0.78
300	300	0.95	0.50	0.99	0.88	0.79	0.86	0.91	0.94	0.95	0.92
300	300	0.95	0.75	0.99	0.87	0.78	0.88	0.94	0.97	0.99	0.94
300	300	0.95	0.85	0.98	0.86	0.78	0.88	0.95	0.99	1.01	0.95
300	300	0.95	0.95	0.98	0.85	0.78	0.91	0.99	1.04	1.06	0.95
300	300	1.00	0.50	0.98	0.88	0.79	0.87	0.92	0.95	0.97	0.95
300	300	1.00	0.75	0.99	0.87	0.77	0.86	0.92	0.96	0.98	0.95
300	300	1.00	0.85	0.99	0.87	0.78	0.88	0.95	0.99	1.02	0.95
300	300	1.00	0.95	0.98	0.85	0.77	0.91	1.01	1.08	1.11	1.00
300	300	1.00	1.00	0.98	0.83	0.75	0.90	1.00	1.05	1.08	0.96

Ratio between the MSE for the estimated IRFs obtained by fitting a VECM on $\widehat{\mathbf{F}}_t$, and the MSE for the estimated IRFs obtained by fitting a VECM on the common factors estimated as in Bai and Ng (2004). Values smaller than one indicate a better performance of our method. T is the number of observations, n is the number of variables, and $n_1 = \lceil n^{\delta} \rceil$ is the number of $I(1)$ idiosyncratic components, and $n_b = \lceil n^{\eta} \rceil$ is the number of variables with a deterministic linear trend.

Table F6a: MonteCarlo Simulations - Impulse Responses
Mean Squared Errors relative to Bai and Ng (2004)
VECM Estimation – First variable, first shock

T	n	δ	η	$k=0$	$k=1$	$k=4$	$k=8$	$k=12$	$k=16$	$k=20$	$k=100$
75	50	0.50	0.50	1.07	0.98	0.86	0.81	0.78	0.76	0.75	0.73
75	50	0.75	0.50	1.05	0.98	0.98	0.97	0.94	0.91	0.89	0.86
75	50	0.75	0.75	0.94	0.97	0.96	0.96	0.93	0.90	0.88	0.84
75	50	0.85	0.50	1.04	0.95	0.98	0.99	0.97	0.95	0.94	0.91
75	50	0.85	0.75	0.95	0.98	1.06	1.07	1.03	1.00	0.98	0.93
75	50	0.85	0.85	1.06	0.99	1.08	1.10	1.05	1.01	0.99	0.95
75	50	0.95	0.50	1.00	0.94	1.03	1.07	1.05	1.02	1.00	0.97
75	50	0.95	0.75	1.05	0.97	1.00	1.07	1.05	1.01	0.99	0.94
75	50	0.95	0.85	1.04	0.95	1.04	1.11	1.07	1.02	0.99	0.93
75	50	0.95	0.95	1.01	1.00	1.11	1.12	1.05	0.99	0.95	0.88
75	50	1.00	0.50	1.11	0.95	1.00	1.03	1.01	0.98	0.96	0.93
75	50	1.00	0.75	1.01	0.97	1.08	1.13	1.10	1.07	1.04	0.98
75	50	1.00	0.85	1.05	0.96	1.06	1.11	1.08	1.05	1.02	0.96
75	50	1.00	0.95	0.99	0.93	1.06	1.12	1.06	1.00	0.96	0.91
75	50	1.00	1.00	1.01	0.91	1.07	1.11	1.04	0.98	0.94	0.87
100	50	0.50	0.50	0.91	0.94	0.84	0.81	0.78	0.75	0.73	0.70
100	50	0.75	0.50	0.97	0.92	0.87	0.91	0.89	0.87	0.86	0.81
100	50	0.75	0.75	1.00	0.97	0.97	0.99	0.95	0.92	0.90	0.85
100	50	0.85	0.50	1.03	0.92	0.97	1.07	1.06	1.03	1.00	0.94
100	50	0.85	0.75	0.99	0.93	0.96	1.04	1.04	1.03	1.01	0.95
100	50	0.85	0.85	0.91	0.91	0.99	1.08	1.07	1.04	1.02	0.94
100	50	0.95	0.50	0.96	0.91	0.94	1.05	1.06	1.05	1.04	0.99
100	50	0.95	0.75	0.98	0.92	1.01	1.10	1.08	1.05	1.03	0.97
100	50	0.95	0.85	1.00	0.90	1.02	1.11	1.09	1.06	1.02	0.91
100	50	0.95	0.95	1.03	0.88	0.91	1.03	1.02	0.98	0.95	0.85
100	50	1.00	0.50	1.05	0.92	1.00	1.11	1.11	1.08	1.06	0.97
100	50	1.00	0.75	0.99	0.90	0.97	1.07	1.06	1.03	1.00	0.90
100	50	1.00	0.85	1.01	0.91	1.01	1.12	1.10	1.06	1.04	0.95
100	50	1.00	0.95	1.10	0.91	1.04	1.17	1.15	1.10	1.06	0.95
100	50	1.00	1.00	1.01	0.87	0.99	1.11	1.08	1.02	0.98	0.88

Ratio between the MSE for the estimated IRF $\widehat{\phi}_{11}(L)$ obtained by fitting a VECM on $\widehat{\mathbf{F}}_t$, and the MSE for the estimated IRFs obtained by fitting a VECM on the common factors estimated as in Bai and Ng (2004). Values smaller than one indicate a better performance of our method. T is the number of observations, n is the number of variables, and $n_1 = \lceil n^\delta \rceil$ is the number of $I(1)$ idiosyncratic components, and $n_b = \lceil n^\eta \rceil$ is the number of variables with a deterministic linear trend.

Table F6b: MonteCarlo Simulations - Impulse Responses
Mean Squared Errors relative to Bai and Ng (2004)
VECM Estimation – First variable, first shock

T	n	δ	η	$k=0$	$k=1$	$k=4$	$k=8$	$k=12$	$k=16$	$k=20$	$k=100$
75	75	0.50	0.50	0.96	1.03	0.85	0.75	0.70	0.68	0.67	0.65
75	75	0.75	0.50	1.03	1.00	0.91	0.90	0.88	0.86	0.85	0.83
75	75	0.75	0.75	0.98	1.02	0.90	0.85	0.81	0.78	0.77	0.74
75	75	0.85	0.50	0.99	1.00	0.97	0.95	0.92	0.90	0.88	0.88
75	75	0.85	0.75	1.04	1.03	1.00	1.00	0.95	0.92	0.90	0.86
75	75	0.85	0.85	1.11	1.01	1.03	1.05	1.01	0.97	0.95	0.91
75	75	0.95	0.50	0.97	0.96	1.00	1.01	1.00	0.97	0.95	0.91
75	75	0.95	0.75	0.89	0.98	1.00	1.03	1.01	0.97	0.95	0.90
75	75	0.95	0.85	0.98	1.00	1.06	1.11	1.08	1.04	1.00	0.93
75	75	0.95	0.95	0.94	0.99	1.08	1.12	1.06	1.00	0.96	0.90
75	75	1.00	0.50	0.97	0.95	0.95	1.00	1.00	0.98	0.97	0.94
75	75	1.00	0.75	1.04	0.96	1.01	1.06	1.06	1.03	1.01	0.96
75	75	1.00	0.85	1.00	0.98	1.07	1.12	1.08	1.04	1.01	0.96
75	75	1.00	0.95	1.02	0.97	1.09	1.14	1.09	1.03	1.00	0.93
75	75	1.00	1.00	0.99	0.99	1.15	1.14	1.03	0.95	0.91	0.83
100	75	0.50	0.50	0.94	0.95	0.77	0.73	0.69	0.67	0.66	0.63
100	75	0.75	0.50	1.04	0.97	0.87	0.86	0.83	0.81	0.79	0.76
100	75	0.75	0.75	0.97	0.94	0.88	0.87	0.84	0.82	0.80	0.76
100	75	0.85	0.50	1.04	0.95	0.95	0.96	0.95	0.93	0.91	0.86
100	75	0.85	0.75	0.98	0.95	0.95	0.99	0.98	0.95	0.93	0.87
100	75	0.85	0.85	1.01	0.92	0.94	1.00	0.99	0.97	0.94	0.87
100	75	0.95	0.50	0.95	0.94	0.97	1.03	1.02	1.01	1.00	0.95
100	75	0.95	0.75	1.01	0.92	0.97	1.07	1.07	1.05	1.03	0.95
100	75	0.95	0.85	1.03	0.92	0.99	1.08	1.06	1.02	1.00	0.93
100	75	0.95	0.95	1.06	0.95	1.03	1.13	1.12	1.09	1.06	0.97
100	75	1.00	0.50	1.03	0.91	0.94	1.01	1.00	0.99	0.97	0.93
100	75	1.00	0.75	0.95	0.92	0.99	1.11	1.09	1.05	1.02	0.94
100	75	1.00	0.85	1.00	0.90	1.00	1.11	1.10	1.07	1.04	0.93
100	75	1.00	0.95	1.03	0.90	1.06	1.18	1.14	1.08	1.03	0.92
100	75	1.00	1.00	0.90	0.89	1.00	1.10	1.07	1.02	0.98	0.88

Ratio between the MSE for the estimated IRF $\widehat{\phi}_{11}(L)$ obtained by fitting a VECM on $\widehat{\mathbf{F}}_t$, and the MSE for the estimated IRFs obtained by fitting a VECM on the common factors estimated as in Bai and Ng (2004). Values smaller than one indicate a better performance of our method. T is the number of observations, n is the number of variables, and $n_1 = \lceil n^{\delta} \rceil$ is the number of $I(1)$ idiosyncratic components, and $n_b = \lceil n^{\eta} \rceil$ is the number of variables with a deterministic linear trend.

Table F6c: MonteCarlo Simulations - Impulse Responses
Mean Squared Errors relative to Bai and Ng (2004)
VECM Estimation – First variable, first shock

T	n	δ	η	$k=0$	$k=1$	$k=4$	$k=8$	$k=12$	$k=16$	$k=20$	$k=100$
100	100	0.50	0.50	0.95	0.98	0.77	0.69	0.65	0.63	0.62	0.60
100	100	0.75	0.50	1.03	0.98	0.90	0.88	0.83	0.80	0.78	0.74
100	100	0.75	0.75	0.95	0.97	0.87	0.82	0.78	0.75	0.73	0.69
100	100	0.85	0.50	1.02	0.94	0.94	0.96	0.94	0.92	0.91	0.86
100	100	0.85	0.75	1.03	0.95	0.95	0.98	0.95	0.92	0.90	0.83
100	100	0.85	0.85	1.05	0.94	0.95	1.01	1.01	0.99	0.97	0.90
100	100	0.95	0.50	0.94	0.94	0.91	0.97	0.98	0.98	0.96	0.91
100	100	0.95	0.75	1.02	0.93	0.94	1.00	1.00	0.98	0.97	0.91
100	100	0.95	0.85	1.05	0.94	1.01	1.12	1.11	1.08	1.05	0.95
100	100	0.95	0.95	0.95	0.93	1.04	1.12	1.09	1.05	1.01	0.91
100	100	1.00	0.50	0.99	0.93	0.96	1.05	1.05	1.02	1.00	0.94
100	100	1.00	0.75	1.03	0.95	1.03	1.12	1.11	1.09	1.06	0.99
100	100	1.00	0.85	1.16	0.93	1.05	1.12	1.09	1.05	1.02	0.94
100	100	1.00	0.95	1.04	0.92	1.03	1.12	1.09	1.04	1.00	0.89
100	100	1.00	1.00	1.05	0.94	1.11	1.18	1.13	1.07	1.03	0.92
200	200	0.50	0.50	0.94	0.89	0.72	0.67	0.62	0.59	0.57	0.53
200	200	0.75	0.50	1.09	0.88	0.77	0.78	0.75	0.73	0.72	0.67
200	200	0.75	0.75	1.00	0.88	0.75	0.73	0.69	0.67	0.65	0.59
200	200	0.85	0.50	0.96	0.82	0.78	0.84	0.84	0.82	0.81	0.73
200	200	0.85	0.75	0.94	0.82	0.79	0.86	0.86	0.85	0.84	0.75
200	200	0.85	0.85	0.97	0.84	0.80	0.91	0.92	0.92	0.90	0.79
200	200	0.95	0.50	1.04	0.82	0.79	0.94	0.98	0.99	0.98	0.92
200	200	0.95	0.75	0.95	0.83	0.81	0.99	1.04	1.06	1.05	0.95
200	200	0.95	0.85	1.05	0.81	0.82	1.07	1.13	1.14	1.12	0.96
200	200	0.95	0.95	0.93	0.79	0.80	1.03	1.09	1.09	1.08	0.92
200	200	1.00	0.50	0.98	0.82	0.81	1.00	1.04	1.06	1.05	0.98
200	200	1.00	0.75	0.97	0.81	0.82	1.00	1.06	1.06	1.06	0.95
200	200	1.00	0.85	0.96	0.80	0.80	1.00	1.06	1.06	1.06	0.93
200	200	1.00	0.95	0.94	0.79	0.84	1.12	1.19	1.19	1.17	0.95
200	200	1.00	1.00	1.12	0.76	0.83	1.14	1.18	1.17	1.15	0.96
300	300	0.50	0.50	1.00	0.83	0.66	0.65	0.59	0.55	0.53	0.48
300	300	0.75	0.50	0.96	0.78	0.66	0.70	0.67	0.65	0.63	0.52
300	300	0.75	0.75	1.02	0.74	0.67	0.74	0.72	0.71	0.70	0.62
300	300	0.85	0.50	0.99	0.75	0.73	0.83	0.84	0.84	0.84	0.77
300	300	0.85	0.75	0.91	0.77	0.72	0.84	0.85	0.85	0.84	0.73
300	300	0.85	0.85	0.98	0.74	0.70	0.84	0.86	0.88	0.89	0.83
300	300	0.95	0.50	0.96	0.76	0.73	0.91	0.97	0.99	1.00	0.92
300	300	0.95	0.75	0.94	0.77	0.73	0.93	0.99	1.01	1.02	0.96
300	300	0.95	0.85	1.00	0.73	0.70	0.94	1.03	1.06	1.07	0.98
300	300	0.95	0.95	1.09	0.73	0.73	0.99	1.08	1.10	1.11	0.94
300	300	1.00	0.50	1.10	0.79	0.72	0.91	0.98	1.00	1.01	0.95
300	300	1.00	0.75	0.94	0.77	0.72	0.94	1.03	1.05	1.06	0.99
300	300	1.00	0.85	0.98	0.78	0.73	0.95	1.02	1.04	1.04	0.92
300	300	1.00	0.95	0.98	0.72	0.71	1.02	1.13	1.16	1.17	1.00
300	300	1.00	1.00	1.01	0.68	0.67	1.02	1.13	1.15	1.16	0.98

Ratio between the MSE for the estimated IRF $\widehat{\phi}_{11}(L)$ obtained by fitting a VECM on $\widehat{\mathbf{F}}_t$, and the MSE for the estimated IRFs obtained by fitting a VECM on the common factors estimated as in Bai and Ng (2004). Values smaller than one indicate a better performance of our method. T is the number of observations, n is the number of variables, and $n_1 = \lceil n^{\delta} \rceil$ is the number of $I(1)$ idiosyncratic components, and $n_b = \lceil n^{\eta} \rceil$ is the number of variables with a deterministic linear trend.

Table F7: MONTECARLO SIMULATIONS - IMPULSE RESPONSES
MEAN SQUARED ERRORS
Unrestricted VAR in Levels Estimation – All variables, All Shocks

T	n	δ	$k=0$	$k=1$	$k=4$	$k=8$	$k=12$	$k=16$	$k=20$	$k=100$
75	50	0.50	0.10	0.11	0.20	0.32	0.42	0.50	0.56	0.77
75	50	0.75	0.10	0.12	0.23	0.36	0.47	0.54	0.59	0.75
75	50	0.85	0.10	0.13	0.25	0.40	0.50	0.58	0.63	0.76
75	50	0.95	0.10	0.14	0.27	0.41	0.51	0.59	0.64	0.74
75	50	1.00	0.10	0.14	0.29	0.45	0.55	0.62	0.68	0.75
100	50	0.50	0.07	0.07	0.14	0.24	0.32	0.40	0.46	0.72
100	50	0.75	0.07	0.09	0.17	0.28	0.37	0.45	0.51	0.74
100	50	0.85	0.07	0.09	0.18	0.29	0.38	0.45	0.52	0.74
100	50	0.95	0.07	0.10	0.22	0.34	0.44	0.51	0.57	0.76
100	50	1.00	0.07	0.11	0.24	0.36	0.45	0.52	0.58	0.74
75	75	0.50	0.09	0.10	0.19	0.32	0.42	0.51	0.56	0.77
75	75	0.75	0.09	0.10	0.20	0.33	0.44	0.52	0.58	0.76
75	75	0.85	0.09	0.11	0.23	0.37	0.48	0.56	0.62	0.77
75	75	0.95	0.09	0.12	0.25	0.40	0.51	0.59	0.65	0.77
75	75	1.00	0.08	0.12	0.25	0.40	0.52	0.61	0.67	0.76
100	75	0.50	0.06	0.07	0.14	0.24	0.33	0.40	0.47	0.74
100	75	0.75	0.06	0.07	0.15	0.26	0.35	0.43	0.49	0.75
100	75	0.85	0.06	0.08	0.17	0.28	0.37	0.45	0.52	0.76
100	75	0.95	0.06	0.09	0.19	0.31	0.41	0.49	0.56	0.76
100	75	1.00	0.06	0.09	0.20	0.32	0.42	0.50	0.57	0.76
100	100	0.50	0.06	0.07	0.13	0.23	0.31	0.39	0.46	0.76
100	100	0.75	0.05	0.07	0.14	0.25	0.35	0.43	0.50	0.76
100	100	0.85	0.05	0.07	0.15	0.26	0.36	0.45	0.51	0.74
100	100	0.95	0.06	0.08	0.18	0.29	0.39	0.48	0.54	0.74
100	100	1.00	0.06	0.08	0.19	0.31	0.40	0.49	0.55	0.74
200	200	0.50	0.02	0.03	0.05	0.10	0.14	0.19	0.24	0.67
200	200	0.75	0.02	0.03	0.06	0.11	0.16	0.22	0.27	0.68
200	200	0.85	0.02	0.03	0.07	0.12	0.17	0.23	0.28	0.70
200	200	0.95	0.02	0.04	0.08	0.14	0.19	0.25	0.31	0.72
200	200	1.00	0.02	0.04	0.09	0.14	0.20	0.26	0.32	0.74
300	300	0.50	0.02	0.02	0.03	0.06	0.08	0.11	0.15	0.58
300	300	0.75	0.02	0.02	0.04	0.07	0.10	0.14	0.17	0.61
300	300	0.85	0.02	0.02	0.05	0.08	0.11	0.15	0.18	0.63
300	300	0.95	0.02	0.03	0.06	0.08	0.12	0.16	0.20	0.67
300	300	1.00	0.02	0.03	0.07	0.10	0.13	0.17	0.20	0.67

MSE for the estimated IRFs by fitting an unrestricted VAR on $\widehat{\mathbf{F}}_t$. T is the number of observations, n is the number of variables, and $n_1 = \lceil n^\delta \rceil$ is the number of $I(1)$ idiosyncratic components. In these simulations there are $n_b = \lceil n^\eta \rceil$ variables with a deterministic linear trend, with $n_b = n_1$.

Table F8a: MonteCarlo Simulations - Impulse Responses
Mean Squared Errors
Unrestricted VAR in Levels Estimation – All variables, All Shocks

T	n	δ	η	$k=0$	$k=1$	$k=4$	$k=8$	$k=12$	$k=16$	$k=20$	$k=100$
75	50	0.50	0.50	0.21	0.19	0.30	0.41	0.49	0.55	0.60	0.77
75	50	0.75	0.50	0.42	0.43	0.57	0.58	0.61	0.65	0.68	0.77
75	50	0.75	0.75	0.22	0.20	0.33	0.46	0.55	0.62	0.66	0.76
75	50	0.85	0.50	0.25	0.24	0.36	0.47	0.56	0.62	0.67	0.76
75	50	0.85	0.75	0.35	0.30	0.39	0.48	0.57	0.63	0.68	0.76
75	50	0.85	0.85	0.26	0.25	0.39	0.51	0.59	0.65	0.69	0.76
75	50	0.95	0.50	0.25	0.25	0.39	0.52	0.60	0.65	0.69	0.75
75	50	0.95	0.75	0.29	0.32	0.47	0.58	0.66	0.71	0.75	0.78
75	50	0.95	0.85	0.31	0.30	0.42	0.54	0.63	0.68	0.72	0.76
75	50	0.95	0.95	0.44	0.39	0.53	0.62	0.67	0.70	0.73	0.77
75	50	1.00	0.50	0.26	0.27	0.43	0.54	0.62	0.68	0.72	0.76
75	50	1.00	0.75	0.24	0.28	0.45	0.56	0.63	0.69	0.72	0.76
75	50	1.00	0.85	0.19	0.23	0.38	0.52	0.62	0.68	0.73	0.77
75	50	1.00	0.95	0.23	0.26	0.44	0.58	0.66	0.71	0.74	0.77
75	50	1.00	1.00	0.27	0.26	0.40	0.54	0.63	0.69	0.72	0.77
100	50	0.50	0.50	0.11	0.11	0.19	0.29	0.38	0.45	0.51	0.74
100	50	0.75	0.50	0.17	0.15	0.27	0.37	0.44	0.51	0.56	0.76
100	50	0.75	0.75	0.14	0.14	0.25	0.36	0.45	0.52	0.57	0.75
100	50	0.85	0.50	0.14	0.15	0.27	0.37	0.45	0.52	0.57	0.73
100	50	0.85	0.75	0.11	0.12	0.25	0.37	0.47	0.54	0.60	0.77
100	50	0.85	0.85	0.15	0.15	0.28	0.40	0.50	0.58	0.64	0.77
100	50	0.95	0.50	0.12	0.15	0.28	0.41	0.50	0.57	0.62	0.76
100	50	0.95	0.75	0.14	0.17	0.30	0.42	0.52	0.60	0.65	0.78
100	50	0.95	0.85	0.12	0.15	0.28	0.41	0.51	0.58	0.64	0.78
100	50	0.95	0.95	0.14	0.16	0.29	0.41	0.51	0.58	0.63	0.75
100	50	1.00	0.50	0.14	0.18	0.31	0.43	0.52	0.59	0.64	0.76
100	50	1.00	0.75	0.11	0.14	0.27	0.40	0.49	0.56	0.62	0.75
100	50	1.00	0.85	0.15	0.18	0.32	0.45	0.54	0.60	0.65	0.75
100	50	1.00	0.95	0.15	0.17	0.31	0.44	0.55	0.62	0.67	0.77
100	50	1.00	1.00	0.15	0.17	0.31	0.43	0.53	0.59	0.64	0.77

MSE for the estimated IRFs by fitting an unrestricted VAR on $\widehat{\mathbf{F}}_t$. T is the number of observations, n is the number of variables, and $n_1 = \lceil n^\delta \rceil$ is the number of $I(1)$ idiosyncratic components, and $n_b = \lceil n^\eta \rceil$ is the number of variables with a deterministic linear trend.

Table F8b: MonteCarlo Simulations - Impulse Responses
Mean Squared Errors
Unrestricted VAR in Levels Estimation – All variables, All Shocks

T	n	δ	η	$k=0$	$k=1$	$k=4$	$k=8$	$k=12$	$k=16$	$k=20$	$k=100$
75	75	0.50	0.50	0.19	0.16	0.25	0.36	0.45	0.52	0.57	0.75
75	75	0.75	0.50	0.22	0.21	0.33	0.41	0.50	0.56	0.62	0.76
75	75	0.75	0.75	0.20	0.20	0.32	0.42	0.52	0.59	0.64	0.77
75	75	0.85	0.50	0.29	0.25	0.36	0.45	0.54	0.61	0.66	0.77
75	75	0.85	0.75	0.23	0.22	0.35	0.45	0.55	0.62	0.67	0.76
75	75	0.85	0.85	0.24	0.19	0.33	0.47	0.57	0.64	0.69	0.77
75	75	0.95	0.50	0.20	0.21	0.33	0.46	0.56	0.64	0.69	0.77
75	75	0.95	0.75	0.23	0.24	0.37	0.48	0.58	0.65	0.70	0.76
75	75	0.95	0.85	0.15	0.16	0.32	0.47	0.58	0.66	0.71	0.77
75	75	0.95	0.95	0.17	0.18	0.33	0.49	0.61	0.68	0.71	0.76
75	75	1.00	0.50	0.17	0.20	0.34	0.47	0.57	0.63	0.68	0.75
75	75	1.00	0.75	0.19	0.21	0.35	0.48	0.58	0.66	0.70	0.74
75	75	1.00	0.85	0.17	0.19	0.35	0.50	0.62	0.69	0.74	0.78
75	75	1.00	0.95	0.13	0.16	0.31	0.48	0.61	0.68	0.72	0.75
75	75	1.00	1.00	0.18	0.21	0.34	0.50	0.61	0.67	0.70	0.75
100	75	0.50	0.50	0.09	0.10	0.18	0.27	0.36	0.43	0.49	0.76
100	75	0.75	0.50	0.10	0.11	0.20	0.30	0.39	0.46	0.53	0.75
100	75	0.75	0.75	0.11	0.12	0.21	0.32	0.42	0.50	0.56	0.76
100	75	0.85	0.50	0.09	0.10	0.20	0.31	0.41	0.48	0.54	0.74
100	75	0.85	0.75	0.14	0.14	0.23	0.34	0.43	0.50	0.56	0.74
100	75	0.85	0.85	0.11	0.12	0.22	0.34	0.45	0.53	0.59	0.76
100	75	0.95	0.50	0.09	0.12	0.24	0.35	0.45	0.53	0.59	0.77
100	75	0.95	0.75	0.17	0.18	0.30	0.39	0.48	0.55	0.61	0.75
100	75	0.95	0.85	0.12	0.13	0.25	0.38	0.48	0.56	0.62	0.76
100	75	0.95	0.95	0.09	0.11	0.23	0.37	0.48	0.56	0.62	0.76
100	75	1.00	0.50	0.11	0.14	0.26	0.38	0.47	0.55	0.61	0.77
100	75	1.00	0.75	0.13	0.16	0.29	0.40	0.50	0.58	0.64	0.76
100	75	1.00	0.85	0.11	0.13	0.26	0.40	0.50	0.59	0.65	0.78
100	75	1.00	0.95	0.11	0.13	0.25	0.39	0.49	0.58	0.63	0.76
100	75	1.00	1.00	0.09	0.12	0.24	0.39	0.50	0.58	0.64	0.75

MSE for the estimated IRFs by fitting an unrestricted VAR on $\widehat{\mathbf{F}}_t$. T is the number of observations, n is the number of variables, and $n_1 = \lceil n^\delta \rceil$ is the number of $I(1)$ idiosyncratic components, and $n_b = \lceil n^\eta \rceil$ is the number of variables with a deterministic linear trend.

Table F8c: MonteCarlo Simulations - Impulse Responses
Mean Squared Errors
Unrestricted VAR in Levels Estimation – All variables, All Shocks

T	n	δ	η	$k=0$	$k=1$	$k=4$	$k=8$	$k=12$	$k=16$	$k=20$	$k=100$
100	100	0.50	0.50	0.09	0.10	0.17	0.26	0.35	0.42	0.48	0.75
100	100	0.75	0.50	0.09	0.09	0.18	0.28	0.36	0.44	0.50	0.74
100	100	0.75	0.75	0.09	0.10	0.20	0.31	0.41	0.49	0.55	0.77
100	100	0.85	0.50	0.12	0.13	0.23	0.32	0.41	0.48	0.55	0.75
100	100	0.85	0.75	0.10	0.11	0.21	0.32	0.42	0.50	0.57	0.77
100	100	0.85	0.85	0.10	0.11	0.21	0.33	0.43	0.51	0.57	0.75
100	100	0.95	0.50	0.09	0.10	0.21	0.33	0.43	0.51	0.58	0.77
100	100	0.95	0.75	0.08	0.10	0.20	0.33	0.43	0.52	0.58	0.74
100	100	0.95	0.85	0.09	0.11	0.23	0.35	0.46	0.55	0.61	0.76
100	100	0.95	0.95	0.09	0.10	0.21	0.35	0.47	0.57	0.63	0.76
100	100	1.00	0.50	0.08	0.10	0.21	0.34	0.43	0.51	0.57	0.75
100	100	1.00	0.75	0.09	0.11	0.22	0.36	0.47	0.55	0.61	0.76
100	100	1.00	0.85	0.08	0.10	0.21	0.34	0.45	0.53	0.60	0.75
100	100	1.00	0.95	0.08	0.10	0.22	0.38	0.50	0.58	0.64	0.76
100	100	1.00	1.00	0.08	0.11	0.21	0.36	0.46	0.54	0.60	0.73
200	200	0.50	0.50	0.04	0.04	0.07	0.12	0.17	0.21	0.26	0.68
200	200	0.75	0.50	0.03	0.04	0.08	0.12	0.17	0.22	0.27	0.68
200	200	0.75	0.75	0.03	0.04	0.08	0.13	0.18	0.24	0.30	0.71
200	200	0.85	0.50	0.03	0.04	0.09	0.14	0.19	0.24	0.29	0.70
200	200	0.85	0.75	0.03	0.04	0.09	0.14	0.20	0.26	0.31	0.71
200	200	0.85	0.85	0.03	0.04	0.08	0.14	0.20	0.26	0.32	0.72
200	200	0.95	0.50	0.03	0.04	0.09	0.15	0.20	0.26	0.31	0.72
200	200	0.95	0.75	0.03	0.05	0.09	0.15	0.21	0.27	0.33	0.74
200	200	0.95	0.85	0.03	0.04	0.09	0.15	0.22	0.28	0.34	0.72
200	200	0.95	0.95	0.03	0.04	0.09	0.16	0.23	0.30	0.37	0.74
200	200	1.00	0.50	0.03	0.05	0.10	0.15	0.21	0.27	0.32	0.72
200	200	1.00	0.75	0.03	0.04	0.10	0.16	0.22	0.28	0.34	0.74
200	200	1.00	0.85	0.03	0.05	0.11	0.17	0.24	0.31	0.37	0.75
200	200	1.00	0.95	0.03	0.05	0.11	0.17	0.25	0.32	0.38	0.76
200	200	1.00	1.00	0.03	0.04	0.10	0.17	0.24	0.31	0.37	0.72
300	300	0.50	0.50	0.02	0.02	0.04	0.07	0.10	0.13	0.16	0.58
300	300	0.75	0.50	0.02	0.02	0.05	0.08	0.11	0.14	0.18	0.60
300	300	0.75	0.75	0.02	0.02	0.05	0.08	0.11	0.15	0.18	0.61
300	300	0.85	0.50	0.02	0.03	0.05	0.08	0.12	0.15	0.19	0.62
300	300	0.85	0.75	0.02	0.03	0.05	0.09	0.12	0.16	0.20	0.64
300	300	0.85	0.85	0.02	0.03	0.05	0.09	0.13	0.17	0.21	0.67
300	300	0.95	0.50	0.02	0.03	0.06	0.09	0.13	0.16	0.21	0.66
300	300	0.95	0.75	0.02	0.03	0.06	0.09	0.13	0.17	0.21	0.67
300	300	0.95	0.85	0.02	0.03	0.06	0.10	0.14	0.18	0.23	0.69
300	300	0.95	0.95	0.02	0.03	0.06	0.10	0.15	0.19	0.24	0.69
300	300	1.00	0.50	0.02	0.03	0.07	0.10	0.13	0.17	0.21	0.66
300	300	1.00	0.75	0.02	0.03	0.07	0.10	0.14	0.18	0.22	0.68
300	300	1.00	0.85	0.02	0.03	0.07	0.10	0.14	0.18	0.23	0.69
300	300	1.00	0.95	0.02	0.03	0.07	0.10	0.15	0.19	0.24	0.70
300	300	1.00	1.00	0.02	0.03	0.07	0.10	0.15	0.20	0.25	0.69

MSE for the estimated IRFs by fitting an unrestricted VAR on $\widehat{\mathbf{F}}_t$. T is the number of observations, n is the number of variables, and $n_1 = \lceil n^\delta \rceil$ is the number of $I(1)$ idiosyncratic components, and $n_b = \lceil n^\eta \rceil$ is the number of variables with a deterministic linear trend.

Table F9a: MonteCarlo Simulations - Impulse Responses
Mean Squared Errors
Unrestricted VAR in Levels Estimation – First variable, first shock

T	n	δ	η	$k=0$	$k=1$	$k=4$	$k=8$	$k=12$	$k=16$	$k=20$	$k=100$
75	50	0.50	0.50	0.00	0.03	0.10	0.19	0.26	0.31	0.35	0.45
75	50	0.75	0.50	0.00	0.03	0.11	0.21	0.29	0.34	0.38	0.45
75	50	0.75	0.75	0.00	0.03	0.11	0.21	0.29	0.35	0.38	0.46
75	50	0.85	0.50	0.00	0.03	0.10	0.20	0.28	0.34	0.38	0.45
75	50	0.85	0.75	0.00	0.03	0.12	0.24	0.33	0.38	0.42	0.47
75	50	0.85	0.85	0.00	0.03	0.13	0.25	0.33	0.38	0.42	0.47
75	50	0.95	0.50	0.00	0.03	0.11	0.22	0.30	0.36	0.40	0.46
75	50	0.95	0.75	0.00	0.03	0.11	0.23	0.32	0.38	0.42	0.46
75	50	0.95	0.85	0.00	0.03	0.13	0.25	0.34	0.39	0.43	0.47
75	50	0.95	0.95	0.00	0.03	0.13	0.25	0.34	0.39	0.42	0.46
75	50	1.00	0.50	0.00	0.03	0.12	0.25	0.34	0.41	0.45	0.50
75	50	1.00	0.75	0.00	0.03	0.12	0.23	0.32	0.37	0.41	0.46
75	50	1.00	0.85	0.00	0.04	0.13	0.25	0.34	0.39	0.43	0.47
75	50	1.00	0.95	0.00	0.03	0.12	0.25	0.34	0.39	0.42	0.46
75	50	1.00	1.00	0.00	0.03	0.13	0.26	0.34	0.39	0.42	0.46
100	50	0.50	0.50	0.00	0.02	0.08	0.16	0.22	0.28	0.32	0.47
100	50	0.75	0.50	0.00	0.03	0.08	0.16	0.22	0.28	0.32	0.45
100	50	0.75	0.75	0.00	0.03	0.09	0.18	0.25	0.31	0.35	0.47
100	50	0.85	0.50	0.00	0.02	0.08	0.16	0.23	0.28	0.32	0.44
100	50	0.85	0.75	0.00	0.02	0.09	0.18	0.25	0.31	0.36	0.48
100	50	0.85	0.85	0.00	0.03	0.10	0.20	0.29	0.35	0.40	0.50
100	50	0.95	0.50	0.00	0.03	0.09	0.18	0.25	0.31	0.35	0.45
100	50	0.95	0.75	0.00	0.03	0.10	0.20	0.28	0.35	0.39	0.49
100	50	0.95	0.85	0.00	0.02	0.09	0.19	0.27	0.33	0.38	0.46
100	50	0.95	0.95	0.00	0.03	0.09	0.19	0.27	0.32	0.36	0.45
100	50	1.00	0.50	0.00	0.03	0.09	0.18	0.26	0.32	0.36	0.46
100	50	1.00	0.75	0.00	0.03	0.09	0.18	0.26	0.32	0.36	0.46
100	50	1.00	0.85	0.00	0.03	0.09	0.19	0.27	0.33	0.38	0.46
100	50	1.00	0.95	0.00	0.03	0.10	0.21	0.29	0.35	0.39	0.47
100	50	1.00	1.00	0.00	0.03	0.10	0.21	0.29	0.35	0.39	0.48

MSE for the estimated IRF $\widehat{\phi}_{11}(L)$ by fitting an unrestricted VAR on $\widehat{\mathbf{F}}_t$. T is the number of observations, n is the number of variables, and $n_1 = \lceil n^{\delta} \rceil$ is the number of $I(1)$ idiosyncratic components, and $n_b = \lceil n^{\eta} \rceil$ is the number of variables with a deterministic linear trend.

Table F9b: MonteCarlo Simulations - Impulse Responses
Mean Squared Errors
Unrestricted VAR in Levels Estimation – First variable, first shock

T	n	δ	η	$k=0$	$k=1$	$k=4$	$k=8$	$k=12$	$k=16$	$k=20$	$k=100$
75	75	0.50	0.50	0.00	0.03	0.10	0.19	0.26	0.31	0.35	0.46
75	75	0.75	0.50	0.00	0.03	0.10	0.19	0.26	0.31	0.35	0.45
75	75	0.75	0.75	0.00	0.03	0.11	0.21	0.30	0.35	0.39	0.47
75	75	0.85	0.50	0.00	0.03	0.10	0.21	0.30	0.36	0.40	0.48
75	75	0.85	0.75	0.00	0.03	0.11	0.24	0.33	0.39	0.42	0.49
75	75	0.85	0.85	0.00	0.03	0.12	0.24	0.32	0.38	0.41	0.47
75	75	0.95	0.50	0.00	0.03	0.11	0.22	0.30	0.36	0.39	0.45
75	75	0.95	0.75	0.00	0.03	0.12	0.24	0.33	0.40	0.43	0.48
75	75	0.95	0.85	0.00	0.03	0.11	0.22	0.30	0.35	0.38	0.43
75	75	0.95	0.95	0.00	0.03	0.13	0.26	0.35	0.40	0.43	0.46
75	75	1.00	0.50	0.00	0.03	0.11	0.22	0.30	0.36	0.40	0.46
75	75	1.00	0.75	0.00	0.03	0.12	0.24	0.33	0.39	0.42	0.47
75	75	1.00	0.85	0.00	0.03	0.12	0.25	0.34	0.40	0.43	0.47
75	75	1.00	0.95	0.00	0.03	0.13	0.26	0.34	0.39	0.42	0.45
75	75	1.00	1.00	0.00	0.03	0.13	0.27	0.36	0.41	0.43	0.47
100	75	0.50	0.50	0.00	0.02	0.06	0.13	0.19	0.24	0.28	0.44
100	75	0.75	0.50	0.00	0.02	0.08	0.16	0.23	0.29	0.33	0.47
100	75	0.75	0.75	0.00	0.02	0.08	0.16	0.24	0.29	0.34	0.46
100	75	0.85	0.50	0.00	0.02	0.08	0.16	0.23	0.29	0.33	0.45
100	75	0.85	0.75	0.00	0.02	0.08	0.17	0.25	0.31	0.35	0.47
100	75	0.85	0.85	0.00	0.02	0.08	0.18	0.26	0.32	0.37	0.47
100	75	0.95	0.50	0.00	0.02	0.08	0.16	0.24	0.30	0.35	0.46
100	75	0.95	0.75	0.00	0.02	0.09	0.18	0.26	0.32	0.36	0.46
100	75	0.95	0.85	0.00	0.02	0.09	0.18	0.25	0.31	0.35	0.44
100	75	0.95	0.95	0.00	0.02	0.10	0.20	0.28	0.34	0.38	0.47
100	75	1.00	0.50	0.00	0.03	0.08	0.16	0.23	0.29	0.33	0.46
100	75	1.00	0.75	0.00	0.02	0.08	0.18	0.26	0.31	0.36	0.46
100	75	1.00	0.85	0.00	0.02	0.08	0.17	0.25	0.32	0.36	0.45
100	75	1.00	0.95	0.00	0.02	0.09	0.20	0.29	0.35	0.39	0.47
100	75	1.00	1.00	0.00	0.02	0.09	0.19	0.27	0.32	0.36	0.45

MSE for the estimated IRF $\widehat{\phi}_{11}(L)$ by fitting an unrestricted VAR on $\widehat{\mathbf{F}}_t$. T is the number of observations, n is the number of variables, and $n_1 = \lceil n^{\delta} \rceil$ is the number of $I(1)$ idiosyncratic components, and $n_b = \lceil n^{\eta} \rceil$ is the number of variables with a deterministic linear trend.

Table F9c: MonteCarlo Simulations - Impulse Responses
Mean Squared Errors
Unrestricted VAR in Levels Estimation – First variable, first shock

T	n	δ	η	$k=0$	$k=1$	$k=4$	$k=8$	$k=12$	$k=16$	$k=20$	$k=100$
100	100	0.50	0.50	0.00	0.02	0.07	0.14	0.20	0.25	0.29	0.46
100	100	0.75	0.50	0.00	0.02	0.07	0.14	0.20	0.26	0.30	0.45
100	100	0.75	0.75	0.00	0.02	0.08	0.15	0.22	0.28	0.32	0.46
100	100	0.85	0.50	0.00	0.02	0.07	0.14	0.21	0.27	0.31	0.45
100	100	0.85	0.75	0.00	0.02	0.08	0.16	0.23	0.29	0.33	0.45
100	100	0.85	0.85	0.00	0.02	0.08	0.17	0.25	0.31	0.35	0.46
100	100	0.95	0.50	0.00	0.02	0.07	0.15	0.23	0.29	0.34	0.46
100	100	0.95	0.75	0.00	0.02	0.07	0.17	0.25	0.31	0.35	0.46
100	100	0.95	0.85	0.00	0.02	0.08	0.18	0.27	0.33	0.38	0.47
100	100	0.95	0.95	0.00	0.02	0.09	0.20	0.29	0.35	0.39	0.48
100	100	1.00	0.50	0.00	0.02	0.08	0.16	0.24	0.29	0.34	0.44
100	100	1.00	0.75	0.00	0.02	0.09	0.19	0.27	0.33	0.38	0.47
100	100	1.00	0.85	0.00	0.02	0.09	0.19	0.28	0.34	0.38	0.48
100	100	1.00	0.95	0.00	0.02	0.09	0.20	0.29	0.35	0.40	0.48
100	100	1.00	1.00	0.00	0.02	0.10	0.21	0.30	0.36	0.40	0.48
200	200	0.50	0.50	0.00	0.01	0.03	0.05	0.08	0.12	0.15	0.40
200	200	0.75	0.50	0.00	0.01	0.03	0.06	0.10	0.13	0.16	0.41
200	200	0.75	0.75	0.00	0.01	0.03	0.06	0.10	0.14	0.18	0.43
200	200	0.85	0.50	0.00	0.01	0.03	0.07	0.11	0.15	0.18	0.45
200	200	0.85	0.75	0.00	0.01	0.03	0.07	0.11	0.15	0.19	0.45
200	200	0.85	0.85	0.00	0.01	0.03	0.07	0.11	0.15	0.19	0.42
200	200	0.95	0.50	0.00	0.01	0.03	0.07	0.11	0.15	0.19	0.44
200	200	0.95	0.75	0.00	0.01	0.03	0.07	0.11	0.15	0.19	0.42
200	200	0.95	0.85	0.00	0.01	0.03	0.08	0.13	0.18	0.22	0.47
200	200	0.95	0.95	0.00	0.01	0.03	0.08	0.13	0.18	0.23	0.47
200	200	1.00	0.50	0.00	0.01	0.03	0.07	0.11	0.15	0.19	0.42
200	200	1.00	0.75	0.00	0.01	0.04	0.07	0.11	0.15	0.19	0.43
200	200	1.00	0.85	0.00	0.01	0.03	0.07	0.12	0.17	0.21	0.44
200	200	1.00	0.95	0.00	0.01	0.04	0.08	0.14	0.19	0.23	0.44
200	200	1.00	1.00	0.00	0.01	0.04	0.08	0.13	0.18	0.22	0.44
300	300	0.50	0.50	0.00	0.01	0.02	0.03	0.05	0.07	0.09	0.35
300	300	0.75	0.50	0.00	0.01	0.02	0.03	0.06	0.08	0.10	0.35
300	300	0.75	0.75	0.00	0.01	0.02	0.04	0.06	0.08	0.11	0.38
300	300	0.85	0.50	0.00	0.01	0.02	0.04	0.06	0.09	0.11	0.41
300	300	0.85	0.75	0.00	0.01	0.02	0.04	0.07	0.09	0.12	0.40
300	300	0.85	0.85	0.00	0.01	0.02	0.04	0.07	0.10	0.13	0.42
300	300	0.95	0.50	0.00	0.01	0.02	0.04	0.07	0.10	0.12	0.41
300	300	0.95	0.75	0.00	0.01	0.02	0.04	0.07	0.10	0.13	0.43
300	300	0.95	0.85	0.00	0.01	0.02	0.04	0.07	0.11	0.14	0.42
300	300	0.95	0.95	0.00	0.01	0.02	0.05	0.08	0.12	0.15	0.43
300	300	1.00	0.50	0.00	0.01	0.02	0.04	0.06	0.09	0.12	0.39
300	300	1.00	0.75	0.00	0.01	0.02	0.05	0.07	0.10	0.13	0.43
300	300	1.00	0.85	0.00	0.01	0.02	0.05	0.08	0.11	0.14	0.44
300	300	1.00	0.95	0.00	0.01	0.02	0.05	0.09	0.12	0.16	0.45
300	300	1.00	1.00	0.00	0.01	0.02	0.04	0.08	0.11	0.14	0.40

MSE for the estimated IRF $\widehat{\phi}_{11}(L)$ by fitting an unrestricted VAR on $\widehat{\mathbf{F}}_t$. T is the number of observations, n is the number of variables, and $n_1 = \lceil n^\delta \rceil$ is the number of $I(1)$ idiosyncratic components, and $n_b = \lceil n^\eta \rceil$ is the number of variables with a deterministic linear trend.

Table F10: MonteCarlo Simulations - Impulse Responses
Mean Squared Errors relative to Bai and Ng (2004)
Unrestricted VAR in Levels Estimation – All variables, All Shocks

T	n	δ	$k=0$	$k=1$	$k=4$	$k=8$	$k=12$	$k=16$	$k=20$	$k=100$
75	50	0.50	0.99	0.97	0.91	0.89	0.90	0.91	0.93	0.99
75	50	0.75	0.99	0.98	0.93	0.92	0.93	0.94	0.94	0.99
75	50	0.85	0.98	0.97	0.94	0.94	0.94	0.94	0.95	1.00
75	50	0.95	1.00	0.98	0.96	0.94	0.95	0.96	0.96	0.99
75	50	1.00	1.01	1.00	0.97	0.95	0.96	0.96	0.96	1.00
100	50	0.50	0.99	0.96	0.91	0.91	0.92	0.93	0.94	0.99
100	50	0.75	1.00	0.98	0.92	0.93	0.93	0.94	0.94	0.99
100	50	0.85	1.00	0.98	0.94	0.92	0.93	0.94	0.94	0.99
100	50	0.95	1.00	0.99	0.96	0.95	0.94	0.95	0.95	1.00
100	50	1.00	0.99	0.99	0.97	0.96	0.95	0.95	0.95	0.99
75	75	0.50	0.99	0.98	0.90	0.89	0.90	0.92	0.93	0.99
75	75	0.75	1.00	0.98	0.91	0.90	0.92	0.93	0.94	1.00
75	75	0.85	1.00	0.98	0.93	0.92	0.93	0.94	0.94	1.00
75	75	0.95	0.98	0.98	0.94	0.92	0.93	0.94	0.94	0.99
75	75	1.00	0.99	0.99	0.96	0.93	0.93	0.94	0.95	1.00
100	75	0.50	0.99	0.95	0.90	0.91	0.92	0.93	0.93	0.99
100	75	0.75	1.00	0.98	0.93	0.92	0.92	0.93	0.93	0.99
100	75	0.85	1.00	0.97	0.92	0.91	0.92	0.92	0.93	0.99
100	75	0.95	1.00	0.98	0.94	0.93	0.94	0.94	0.95	0.99
100	75	1.00	1.00	0.99	0.95	0.93	0.93	0.94	0.95	0.99
100	100	0.50	0.99	0.96	0.89	0.88	0.89	0.90	0.92	0.99
100	100	0.75	0.99	0.96	0.92	0.91	0.92	0.93	0.94	0.99
100	100	0.85	0.99	0.97	0.92	0.91	0.93	0.93	0.94	0.99
100	100	0.95	1.00	0.98	0.94	0.92	0.93	0.93	0.94	0.99
100	100	1.00	0.99	0.99	0.96	0.93	0.93	0.93	0.94	0.99
200	200	0.50	0.99	0.93	0.86	0.87	0.89	0.90	0.92	0.98
200	200	0.75	0.99	0.94	0.87	0.90	0.91	0.92	0.93	0.98
200	200	0.85	0.99	0.94	0.90	0.91	0.91	0.92	0.93	0.98
200	200	0.95	0.99	0.96	0.92	0.93	0.93	0.94	0.94	0.99
200	200	1.00	1.00	0.97	0.93	0.93	0.93	0.94	0.94	0.99
300	300	0.50	0.99	0.91	0.83	0.87	0.88	0.90	0.91	0.98
300	300	0.75	0.99	0.91	0.85	0.90	0.91	0.92	0.93	0.98
300	300	0.85	0.99	0.92	0.86	0.90	0.92	0.93	0.94	0.98
300	300	0.95	0.99	0.94	0.88	0.91	0.93	0.94	0.95	0.99
300	300	1.00	0.99	0.95	0.90	0.93	0.94	0.94	0.95	0.99

Ratio between the MSE for the estimated IRFs obtained by fitting an unrestricted VAR on $\widehat{\mathbf{F}}_t$, and the MSE for the estimated IRFs obtained by fitting an unrestricted VAR on the common factors estimated as in Bai and Ng (2004). Values smaller than one indicate a better performance of our method. T is the number of observations, n is the number of variables, and $n_1 = \lceil n^\delta \rceil$ is the number of $I(1)$ idiosyncratic components. In these simulations there are $n_b = \lceil n^\eta \rceil$ variables with a deterministic linear trend, with $n_b = n_1$.

Table F11a: MonteCarlo Simulations - Impulse Responses
Mean Squared Errors relative to Bai and Ng (2004)
Unrestricted VAR in Levels Estimation – All variables, All Shocks

T	n	δ	η	$k=0$	$k=1$	$k=4$	$k=8$	$k=12$	$k=16$	$k=20$	$k=100$
75	50	0.50	0.50	0.93	0.96	0.91	0.91	0.94	0.96	0.97	1.00
75	50	0.75	0.50	1.14	1.15	1.19	1.09	1.03	1.00	1.00	0.99
75	50	0.75	0.75	1.14	1.11	1.05	1.03	1.03	1.02	1.02	1.00
75	50	0.85	0.50	0.95	0.94	0.91	0.93	0.95	0.96	0.97	1.00
75	50	0.85	0.75	0.69	0.64	0.73	0.82	0.94	0.96	0.99	1.01
75	50	0.85	0.85	1.04	1.04	1.03	1.01	1.01	1.02	1.02	1.01
75	50	0.95	0.50	1.05	1.04	0.99	0.99	0.99	0.98	0.98	1.00
75	50	0.95	0.75	0.78	0.80	0.86	0.92	0.96	0.98	0.99	1.00
75	50	0.95	0.85	0.88	0.84	0.84	0.92	0.95	0.98	0.99	1.01
75	50	0.95	0.95	0.92	0.82	0.83	0.94	0.98	1.00	0.99	1.01
75	50	1.00	0.50	0.95	0.95	1.00	0.98	0.97	0.97	0.98	1.00
75	50	1.00	0.75	1.05	1.04	1.02	1.01	1.00	1.00	1.00	1.00
75	50	1.00	0.85	0.98	0.97	0.97	0.99	1.00	1.00	1.00	1.00
75	50	1.00	0.95	0.94	0.94	0.96	1.00	1.02	1.02	1.02	1.00
75	50	1.00	1.00	1.12	1.06	1.00	1.02	1.02	1.01	0.99	1.01
100	50	0.50	0.50	0.98	0.97	0.94	0.94	0.95	0.96	0.96	1.00
100	50	0.75	0.50	1.06	1.02	1.03	0.99	0.97	0.97	0.98	0.99
100	50	0.75	0.75	0.92	0.94	0.95	0.99	1.01	1.01	1.02	1.00
100	50	0.85	0.50	1.01	1.02	0.99	0.97	0.98	0.98	0.99	1.00
100	50	0.85	0.75	0.94	0.93	0.94	0.97	0.99	1.00	1.00	1.00
100	50	0.85	0.85	1.06	1.01	0.96	0.99	1.02	1.03	1.03	1.00
100	50	0.95	0.50	0.98	0.97	0.96	0.95	0.95	0.96	0.97	1.00
100	50	0.95	0.75	0.96	0.96	0.95	0.98	0.99	1.01	1.01	1.00
100	50	0.95	0.85	1.03	1.02	0.99	1.01	1.03	1.03	1.03	1.00
100	50	0.95	0.95	0.83	0.83	0.90	0.99	1.02	1.02	1.02	1.00
100	50	1.00	0.50	1.06	1.06	1.01	0.98	0.98	0.98	0.98	1.00
100	50	1.00	0.75	0.97	0.98	0.97	0.98	0.99	1.00	1.00	1.00
100	50	1.00	0.85	1.01	0.99	1.00	1.01	1.02	1.03	1.02	1.00
100	50	1.00	0.95	1.01	0.95	0.94	1.02	1.05	1.05	1.04	1.01
100	50	1.00	1.00	1.02	0.96	0.97	1.01	1.02	1.02	1.01	1.01

Ratio between the MSE for the estimated IRFs obtained by fitting an unrestricted VAR on $\widehat{\mathbf{F}}_t$, and the MSE for the estimated IRFs obtained by fitting an unrestricted VAR on the common factors estimated as in Bai and Ng (2004). Values smaller than one indicate a better performance of our method. T is the number of observations, n is the number of variables, and $n_1 = \lceil n^\delta \rceil$ is the number of $I(1)$ idiosyncratic components, and $n_b = \lceil n^\eta \rceil$ is the number of variables with a deterministic linear trend.

Table F11b: MonteCarlo Simulations - Impulse Responses
Mean Squared Errors relative to Bai and Ng (2004)
Unrestricted VAR in Levels Estimation – All variables, All Shocks

T	n	δ	η	$k=0$	$k=1$	$k=4$	$k=8$	$k=12$	$k=16$	$k=20$	$k=100$
75	75	0.50	0.50	1.05	1.01	0.94	0.92	0.93	0.95	0.96	1.00
75	75	0.75	0.50	1.02	1.03	1.06	0.97	0.95	0.96	0.96	0.99
75	75	0.75	0.75	1.09	1.10	1.03	1.01	1.01	1.01	1.00	1.00
75	75	0.85	0.50	0.98	1.00	0.94	0.92	0.94	0.95	0.96	1.00
75	75	0.85	0.75	1.09	1.08	1.00	0.97	0.99	1.00	1.00	1.00
75	75	0.85	0.85	1.19	1.07	1.02	1.01	1.03	1.03	1.02	1.00
75	75	0.95	0.50	0.87	0.91	0.97	0.94	0.95	0.95	0.96	1.00
75	75	0.95	0.75	0.96	0.96	0.93	0.96	0.99	1.00	1.00	1.00
75	75	0.95	0.85	0.95	0.93	0.94	0.99	1.02	1.03	1.02	1.00
75	75	0.95	0.95	0.97	0.95	0.97	1.02	1.04	1.04	1.02	1.00
75	75	1.00	0.50	0.90	0.90	0.95	0.95	0.96	0.97	0.97	1.00
75	75	1.00	0.75	1.05	1.04	1.00	0.99	1.01	1.01	1.01	1.00
75	75	1.00	0.85	1.00	0.98	0.98	0.99	1.01	1.02	1.01	1.00
75	75	1.00	0.95	0.99	0.95	0.97	1.02	1.04	1.03	1.02	1.00
75	75	1.00	1.00	1.01	1.00	0.94	1.01	1.03	1.02	1.00	1.01
100	75	0.50	0.50	1.01	0.97	0.94	0.94	0.95	0.96	0.97	0.99
100	75	0.75	0.50	1.01	1.00	0.96	0.95	0.95	0.96	0.96	0.99
100	75	0.75	0.75	1.05	1.04	0.98	0.98	1.00	1.00	1.00	0.99
100	75	0.85	0.50	0.96	0.96	0.93	0.93	0.95	0.96	0.96	0.99
100	75	0.85	0.75	0.96	0.94	0.95	0.97	0.99	1.00	1.00	1.00
100	75	0.85	0.85	0.97	0.95	0.95	1.02	1.04	1.05	1.05	1.00
100	75	0.95	0.50	0.98	0.99	0.97	0.95	0.95	0.96	0.97	1.00
100	75	0.95	0.75	1.36	1.28	1.15	1.05	1.01	0.99	0.99	1.00
100	75	0.95	0.85	1.00	0.99	0.98	1.01	1.02	1.02	1.02	1.00
100	75	0.95	0.95	1.01	0.97	0.98	1.04	1.05	1.06	1.05	1.01
100	75	1.00	0.50	1.09	1.08	1.01	0.98	0.97	0.98	0.98	1.00
100	75	1.00	0.75	1.07	1.02	0.97	0.98	0.99	1.00	1.00	1.00
100	75	1.00	0.85	0.99	0.98	0.97	0.99	1.02	1.02	1.02	1.00
100	75	1.00	0.95	0.99	0.96	0.96	1.00	1.03	1.04	1.04	1.00
100	75	1.00	1.00	1.01	0.97	0.98	1.04	1.06	1.06	1.04	1.00

Ratio between the MSE for the estimated IRFs obtained by fitting an unrestricted VAR on $\widehat{\mathbf{F}}_t$, and the MSE for the estimated IRFs obtained by fitting an unrestricted VAR on the common factors estimated as in Bai and Ng (2004). Values smaller than one indicate a better performance of our method. T is the number of observations, n is the number of variables, and $n_1 = \lceil n^{\delta} \rceil$ is the number of $I(1)$ idiosyncratic components, and $n_b = \lceil n^{\eta} \rceil$ is the number of variables with a deterministic linear trend.

Table F11c: MonteCarlo Simulations - Impulse Responses
Mean Squared Errors relative to Bai and Ng (2004)
Unrestricted VAR in Levels Estimation – All variables, All Shocks

T	n	δ	η	$k=0$	$k=1$	$k=4$	$k=8$	$k=12$	$k=16$	$k=20$	$k=100$
100	100	0.50	0.50	1.00	0.97	0.94	0.94	0.95	0.95	0.96	0.99
100	100	0.75	0.50	1.00	0.98	0.94	0.94	0.94	0.95	0.95	0.99
100	100	0.75	0.75	0.98	0.97	0.95	0.97	0.99	1.00	1.01	1.00
100	100	0.85	0.50	1.06	1.04	0.98	0.95	0.94	0.95	0.96	0.99
100	100	0.85	0.75	0.99	0.97	0.96	0.97	0.99	1.00	1.00	0.99
100	100	0.85	0.85	1.01	1.00	0.98	1.01	1.03	1.04	1.04	1.00
100	100	0.95	0.50	1.02	0.99	0.94	0.94	0.95	0.95	0.96	0.99
100	100	0.95	0.75	1.01	0.98	0.95	0.97	0.98	0.99	1.00	1.00
100	100	0.95	0.85	0.97	0.96	0.96	1.00	1.02	1.03	1.03	1.00
100	100	0.95	0.95	1.02	0.98	0.98	1.04	1.07	1.07	1.06	1.00
100	100	1.00	0.50	0.99	0.98	0.95	0.95	0.96	0.96	0.96	1.00
100	100	1.00	0.75	1.00	0.98	0.96	0.98	0.99	0.99	0.99	1.00
100	100	1.00	0.85	1.00	0.98	0.97	1.00	1.02	1.03	1.03	1.00
100	100	1.00	0.95	0.99	0.97	0.99	1.05	1.07	1.06	1.05	1.00
100	100	1.00	1.00	0.99	0.96	0.97	1.04	1.06	1.06	1.04	1.00
200	200	0.50	0.50	0.99	0.95	0.90	0.92	0.94	0.95	0.96	0.99
200	200	0.75	0.50	0.99	0.95	0.89	0.92	0.94	0.95	0.96	0.99
200	200	0.75	0.75	1.00	0.96	0.91	0.94	0.97	0.99	1.00	1.00
200	200	0.85	0.50	0.99	0.96	0.92	0.93	0.94	0.95	0.96	0.99
200	200	0.85	0.75	1.00	0.96	0.93	0.97	1.00	1.01	1.02	1.00
200	200	0.85	0.85	0.99	0.94	0.90	0.96	1.01	1.04	1.06	1.01
200	200	0.95	0.50	0.99	0.96	0.91	0.93	0.95	0.96	0.97	0.99
200	200	0.95	0.75	0.99	0.96	0.92	0.95	0.98	0.99	1.01	1.00
200	200	0.95	0.85	1.00	0.96	0.92	0.98	1.02	1.04	1.05	1.00
200	200	0.95	0.95	1.00	0.95	0.93	1.03	1.09	1.11	1.12	0.99
200	200	1.00	0.50	1.00	0.97	0.93	0.94	0.96	0.96	0.97	0.99
200	200	1.00	0.75	0.99	0.96	0.93	0.96	0.98	1.00	1.01	1.00
200	200	1.00	0.85	0.99	0.96	0.93	0.98	1.01	1.04	1.05	1.00
200	200	1.00	0.95	0.99	0.95	0.94	1.02	1.08	1.10	1.11	0.99
200	200	1.00	1.00	0.99	0.93	0.92	1.02	1.08	1.10	1.11	0.98
300	300	0.50	0.50	0.99	0.92	0.86	0.89	0.90	0.92	0.93	0.99
300	300	0.75	0.50	0.99	0.92	0.86	0.90	0.92	0.94	0.95	0.99
300	300	0.75	0.75	0.99	0.92	0.87	0.93	0.96	0.98	1.00	1.01
300	300	0.85	0.50	0.99	0.93	0.88	0.92	0.94	0.96	0.96	0.99
300	300	0.85	0.75	0.99	0.93	0.89	0.94	0.98	1.00	1.01	1.02
300	300	0.85	0.85	0.99	0.93	0.89	0.96	1.00	1.04	1.06	1.03
300	300	0.95	0.50	1.00	0.95	0.90	0.93	0.94	0.95	0.96	0.99
300	300	0.95	0.75	0.99	0.94	0.89	0.93	0.97	0.99	1.00	1.01
300	300	0.95	0.85	0.99	0.93	0.90	0.96	1.01	1.03	1.04	1.02
300	300	0.95	0.95	0.99	0.93	0.89	0.98	1.05	1.09	1.11	1.03
300	300	1.00	0.50	0.99	0.95	0.91	0.93	0.94	0.95	0.96	0.99
300	300	1.00	0.75	1.00	0.95	0.90	0.95	0.98	1.00	1.01	1.01
300	300	1.00	0.85	0.99	0.94	0.90	0.95	1.00	1.02	1.04	1.02
300	300	1.00	0.95	0.99	0.92	0.89	0.99	1.06	1.09	1.11	1.02
300	300	1.00	1.00	0.99	0.90	0.86	0.99	1.07	1.12	1.15	1.01

Ratio between the MSE for the estimated IRFs obtained by fitting an unrestricted VAR on $\widehat{\mathbf{F}}_t$, and the MSE for the estimated IRFs obtained by fitting an unrestricted VAR on the common factors estimated as in Bai and Ng (2004). Values smaller than one indicate a better performance of our method. T is the number of observations, n is the number of variables, and $n_1 = \lceil n^{\delta} \rceil$ is the number of $I(1)$ idiosyncratic components, and $n_b = \lceil n^{\eta} \rceil$ is the number of variables with a deterministic linear trend.

Table F12a: MonteCarlo Simulations - Impulse Responses
Mean Squared Errors relative to Bai and Ng (2004)
Unrestricted VAR in Levels Estimation – First Variable, First Shock

T	n	δ	η	$k=0$	$k=1$	$k=4$	$k=8$	$k=12$	$k=16$	$k=20$	$k=100$
T	N	delta	alpha	k=0	k=1	k=4	k=8	k=12	k=16	k=20	k=100
75	50	0.50	0.50	1.07	1.00	0.97	0.98	0.99	0.98	0.98	1.00
75	50	0.75	0.50	1.04	0.98	0.98	1.00	1.00	1.00	0.99	0.99
75	50	0.75	0.75	0.96	0.98	1.00	1.02	1.03	1.02	1.01	1.00
75	50	0.85	0.50	0.98	0.98	1.00	0.99	0.99	0.99	0.98	1.00
75	50	0.85	0.75	0.94	0.99	1.03	1.04	1.03	1.02	1.01	1.00
75	50	0.85	0.85	1.04	0.99	1.05	1.07	1.06	1.05	1.03	1.00
75	50	0.95	0.50	1.03	0.96	0.96	0.98	0.98	0.98	0.98	1.00
75	50	0.95	0.75	1.05	1.00	0.98	1.01	1.02	1.01	1.01	1.00
75	50	0.95	0.85	0.99	1.00	1.06	1.09	1.08	1.06	1.04	1.00
75	50	0.95	0.95	1.03	1.01	1.08	1.11	1.09	1.06	1.03	1.00
75	50	1.00	0.50	0.96	0.97	0.96	0.96	0.97	0.97	0.98	1.00
75	50	1.00	0.75	1.02	1.00	1.01	1.01	1.01	1.00	1.00	1.01
75	50	1.00	0.85	0.93	0.99	1.04	1.07	1.06	1.05	1.03	1.00
75	50	1.00	0.95	1.09	0.98	1.08	1.12	1.10	1.06	1.02	1.00
75	50	1.00	1.00	1.03	0.98	1.10	1.13	1.11	1.06	1.02	0.99
100	50	0.50	0.50	1.04	0.99	0.98	0.98	0.98	0.99	0.99	0.99
100	50	0.75	0.50	0.96	0.98	0.97	0.99	0.99	0.99	0.99	1.00
100	50	0.75	0.75	1.01	1.01	1.02	1.04	1.03	1.03	1.02	1.00
100	50	0.85	0.50	1.01	0.98	0.98	1.00	1.00	1.00	1.00	1.00
100	50	0.85	0.75	1.01	0.98	1.01	1.04	1.04	1.04	1.03	1.00
100	50	0.85	0.85	0.93	0.96	1.03	1.09	1.09	1.08	1.06	1.00
100	50	0.95	0.50	0.96	0.98	0.96	0.99	0.99	1.00	1.00	1.00
100	50	0.95	0.75	1.05	0.99	1.02	1.06	1.05	1.04	1.03	1.00
100	50	0.95	0.85	1.09	0.98	1.02	1.07	1.07	1.06	1.05	1.00
100	50	0.95	0.95	1.04	0.96	1.03	1.11	1.11	1.09	1.06	1.00
100	50	1.00	0.50	1.01	0.99	1.01	1.02	1.02	1.01	1.00	1.00
100	50	1.00	0.75	0.99	0.99	1.01	1.03	1.04	1.03	1.03	1.00
100	50	1.00	0.85	0.97	0.98	1.03	1.07	1.07	1.06	1.04	1.00
100	50	1.00	0.95	1.03	0.97	1.05	1.12	1.11	1.09	1.06	1.00
100	50	1.00	1.00	0.96	0.94	1.06	1.12	1.10	1.07	1.04	1.00

Ratio between the MSE for the estimated IRF $\widehat{\phi}_{11}(L)$ obtained by fitting an unrestricted VAR on $\widehat{\mathbf{F}}_t$, and the MSE for the estimated IRFs obtained by fitting an unrestricted VAR on the common factors estimated as in Bai and Ng (2004). Values smaller than one indicate a better performance of our method. T is the number of observations, n is the number of variables, and $n_1 = \lceil n^\delta \rceil$ is the number of $I(1)$ idiosyncratic components, and $n_b = \lceil n^\eta \rceil$ is the number of variables with a deterministic linear trend.

Table F12b: MonteCarlo Simulations - Impulse Responses
Mean Squared Errors relative to Bai and Ng (2004)
Unrestricted VAR in Levels Estimation – First Variable, First Shock

T	n	δ	η	$k=0$	$k=1$	$k=4$	$k=8$	$k=12$	$k=16$	$k=20$	$k=100$
75	75	0.50	0.50	1.05	1.02	0.97	0.94	0.95	0.95	0.96	0.99
75	75	0.75	0.50	0.95	0.99	0.96	0.96	0.97	0.97	0.97	1.00
75	75	0.75	0.75	0.98	1.01	0.99	1.00	1.01	1.01	1.01	1.00
75	75	0.85	0.50	0.99	0.98	0.93	0.93	0.94	0.96	0.97	1.00
75	75	0.85	0.75	0.97	1.00	0.99	1.01	1.02	1.02	1.01	1.00
75	75	0.85	0.85	0.88	0.99	1.03	1.07	1.06	1.05	1.03	1.00
75	75	0.95	0.50	0.98	0.97	0.96	0.96	0.97	0.97	0.98	1.00
75	75	0.95	0.75	1.04	0.98	0.98	1.00	1.01	1.01	1.00	1.00
75	75	0.95	0.85	0.97	1.01	1.04	1.07	1.06	1.04	1.02	1.00
75	75	0.95	0.95	0.89	1.00	1.09	1.12	1.09	1.06	1.03	1.00
75	75	1.00	0.50	1.01	0.97	0.92	0.95	0.96	0.97	0.97	1.00
75	75	1.00	0.75	1.04	1.00	0.99	1.01	1.02	1.03	1.02	1.00
75	75	1.00	0.85	0.96	0.99	1.02	1.06	1.06	1.05	1.03	1.00
75	75	1.00	0.95	0.87	0.98	1.07	1.11	1.09	1.05	1.02	0.99
75	75	1.00	1.00	1.10	1.02	1.18	1.16	1.11	1.06	1.02	1.01
100	75	0.50	0.50	1.03	1.01	0.94	0.95	0.96	0.97	0.97	0.99
100	75	0.75	0.50	0.97	1.01	0.95	0.96	0.96	0.97	0.97	0.99
100	75	0.75	0.75	0.94	0.98	1.01	1.03	1.03	1.03	1.02	1.00
100	75	0.85	0.50	1.04	0.99	0.98	0.97	0.98	0.98	0.98	1.00
100	75	0.85	0.75	1.05	0.97	1.00	1.01	1.02	1.02	1.01	1.00
100	75	0.85	0.85	1.01	0.98	1.05	1.09	1.09	1.08	1.06	1.00
100	75	0.95	0.50	1.04	0.98	0.97	0.97	0.97	0.97	0.97	1.00
100	75	0.95	0.75	1.01	0.98	1.00	1.02	1.02	1.02	1.01	1.00
100	75	0.95	0.85	1.00	1.00	1.05	1.08	1.08	1.07	1.06	1.00
100	75	0.95	0.95	0.98	1.00	1.06	1.11	1.09	1.07	1.05	1.01
100	75	1.00	0.50	0.98	0.98	0.95	0.96	0.96	0.97	0.97	0.99
100	75	1.00	0.75	1.00	0.98	1.01	1.04	1.04	1.03	1.03	1.00
100	75	1.00	0.85	0.98	0.97	1.02	1.07	1.07	1.07	1.05	1.00
100	75	1.00	0.95	0.99	0.97	1.08	1.13	1.12	1.09	1.06	1.00
100	75	1.00	1.00	1.05	0.98	1.10	1.17	1.15	1.11	1.08	0.99

Ratio between the MSE for the estimated IRF $\widehat{\phi}_{11}(L)$ obtained by fitting an unrestricted VAR on $\widehat{\mathbf{F}}_t$, and the MSE for the estimated IRFs obtained by fitting an unrestricted VAR on the common factors estimated as in Bai and Ng (2004). Values smaller than one indicate a better performance of our method. T is the number of observations, n is the number of variables, and $n_1 = \lceil n^\delta \rceil$ is the number of $I(1)$ idiosyncratic components, and $n_b = \lceil n^\eta \rceil$ is the number of variables with a deterministic linear trend.

Table F12c: MonteCarlo Simulations - Impulse Responses
Mean Squared Errors relative to Bai and Ng (2004)
Unrestricted VAR in Levels Estimation – First Variable, First Shock

T	n	δ	η	$k=0$	$k=1$	$k=4$	$k=8$	$k=12$	$k=16$	$k=20$	$k=100$
100	100	0.50	0.50	0.97	1.00	0.96	0.96	0.96	0.96	0.97	1.00
100	100	0.75	0.50	0.99	1.00	1.00	0.99	0.98	0.98	0.98	0.99
100	100	0.75	0.75	1.00	1.00	1.00	1.01	1.02	1.01	1.01	1.00
100	100	0.85	0.50	1.00	0.97	0.94	0.96	0.97	0.97	0.98	0.99
100	100	0.85	0.75	0.95	0.98	0.99	1.01	1.02	1.02	1.01	1.00
100	100	0.85	0.85	1.02	0.99	1.02	1.06	1.07	1.07	1.06	1.00
100	100	0.95	0.50	1.02	0.99	0.94	0.94	0.96	0.96	0.96	0.99
100	100	0.95	0.75	0.96	0.98	0.98	1.00	1.02	1.02	1.01	1.00
100	100	0.95	0.85	0.96	0.98	1.03	1.07	1.07	1.06	1.05	1.00
100	100	0.95	0.95	0.96	0.97	1.08	1.13	1.11	1.09	1.07	1.00
100	100	1.00	0.50	0.99	0.96	0.94	0.95	0.96	0.97	0.97	1.00
100	100	1.00	0.75	0.99	1.00	1.02	1.03	1.02	1.02	1.01	1.00
100	100	1.00	0.85	0.98	0.97	1.03	1.07	1.07	1.06	1.05	1.00
100	100	1.00	0.95	0.97	0.99	1.08	1.12	1.11	1.09	1.06	1.00
100	100	1.00	1.00	0.97	1.01	1.14	1.19	1.15	1.11	1.08	1.00
200	200	0.50	0.50	0.97	0.96	0.93	0.94	0.95	0.95	0.96	0.99
200	200	0.75	0.50	0.98	0.94	0.93	0.94	0.95	0.96	0.96	0.99
200	200	0.75	0.75	1.00	0.98	0.96	0.99	1.00	1.01	1.01	1.01
200	200	0.85	0.50	1.00	0.95	0.95	0.96	0.96	0.97	0.97	0.99
200	200	0.85	0.75	1.07	0.93	0.96	1.01	1.03	1.03	1.03	1.01
200	200	0.85	0.85	0.98	0.93	0.95	1.05	1.08	1.09	1.09	1.01
200	200	0.95	0.50	1.02	0.93	0.93	0.98	0.98	0.98	0.98	0.99
200	200	0.95	0.75	1.12	0.95	0.94	1.01	1.03	1.04	1.04	1.00
200	200	0.95	0.85	0.98	0.93	0.96	1.07	1.08	1.08	1.08	0.99
200	200	0.95	0.95	0.93	0.91	0.97	1.12	1.15	1.15	1.15	1.00
200	200	1.00	0.50	1.06	0.93	0.93	0.98	0.98	0.98	0.98	0.99
200	200	1.00	0.75	1.01	0.93	0.94	1.00	1.02	1.02	1.02	1.00
200	200	1.00	0.85	0.97	0.92	0.94	1.04	1.06	1.06	1.06	1.00
200	200	1.00	0.95	0.98	0.91	0.97	1.13	1.16	1.16	1.14	1.00
200	200	1.00	1.00	1.03	0.89	1.01	1.17	1.17	1.15	1.13	0.98
300	300	0.50	0.50	1.03	0.95	0.88	0.91	0.92	0.93	0.94	0.99
300	300	0.75	0.50	1.04	0.89	0.90	0.97	0.97	0.97	0.98	0.99
300	300	0.75	0.75	1.00	0.86	0.88	0.98	1.00	1.02	1.02	1.01
300	300	0.85	0.50	1.12	0.89	0.89	0.94	0.95	0.95	0.96	0.99
300	300	0.85	0.75	1.00	0.89	0.91	1.01	1.03	1.04	1.04	1.02
300	300	0.85	0.85	1.02	0.88	0.90	1.03	1.06	1.07	1.08	1.03
300	300	0.95	0.50	1.02	0.91	0.89	0.95	0.96	0.97	0.97	0.99
300	300	0.95	0.75	1.03	0.88	0.88	1.00	1.01	1.02	1.02	1.01
300	300	0.95	0.85	0.97	0.88	0.89	1.04	1.06	1.07	1.07	1.03
300	300	0.95	0.95	1.09	0.86	0.91	1.10	1.14	1.15	1.15	1.02
300	300	1.00	0.50	1.02	0.93	0.90	0.99	1.00	1.00	1.00	0.99
300	300	1.00	0.75	1.00	0.90	0.88	0.98	1.01	1.01	1.01	1.01
300	300	1.00	0.85	1.01	0.90	0.90	1.03	1.06	1.07	1.07	1.02
300	300	1.00	0.95	1.07	0.85	0.88	1.08	1.13	1.14	1.14	1.02
300	300	1.00	1.00	0.97	0.80	0.84	1.11	1.17	1.18	1.18	1.00

Ratio between the MSE for the estimated IRF $\widehat{\phi}_{11}(L)$ obtained by fitting an unrestricted VAR on $\widehat{\mathbf{F}}_t$, and the MSE for the estimated IRFs obtained by fitting an unrestricted VAR on the common factors estimated as in Bai and Ng (2004). Values smaller than one indicate a better performance of our method. T is the number of observations, n is the number of variables, and $n_1 = \lceil n^\delta \rceil$ is the number of $I(1)$ idiosyncratic components, and $n_b = \lceil n^\eta \rceil$ is the number of variables with a deterministic linear trend.

Table F13: MonteCarlo Simulations - Impulse Responses
Mean Squared Errors relative to VAR in Differences
All variables, All Shocks

T	n	δ	$k=0$	$k=1$	$k=4$	$k=8$	$k=12$	$k=16$	$k=20$	$k=100$
75	50	0.50	0.99	0.64	0.41	0.45	0.48	0.49	0.50	0.51
75	50	0.75	1.04	0.67	0.46	0.55	0.63	0.67	0.69	0.72
75	50	0.85	1.00	0.68	0.52	0.66	0.76	0.82	0.85	0.91
75	50	0.95	1.04	0.70	0.55	0.70	0.81	0.88	0.92	1.01
75	50	1.00	1.05	0.72	0.59	0.76	0.89	0.96	1.01	1.09
100	50	0.50	0.99	0.56	0.31	0.34	0.36	0.38	0.38	0.39
100	50	0.75	1.03	0.62	0.38	0.47	0.54	0.59	0.62	0.69
100	50	0.85	1.01	0.63	0.41	0.50	0.60	0.67	0.72	0.82
100	50	0.95	1.00	0.66	0.49	0.63	0.75	0.84	0.90	1.03
100	50	1.00	1.02	0.70	0.52	0.65	0.77	0.86	0.92	1.05
75	75	0.50	1.02	0.65	0.40	0.43	0.44	0.45	0.45	0.45
75	75	0.75	1.01	0.65	0.42	0.50	0.57	0.60	0.61	0.64
75	75	0.85	1.00	0.67	0.48	0.61	0.71	0.76	0.80	0.85
75	75	0.95	1.00	0.69	0.53	0.68	0.79	0.86	0.90	0.96
75	75	1.00	1.01	0.69	0.53	0.71	0.84	0.91	0.95	1.04
100	75	0.50	0.99	0.57	0.31	0.33	0.35	0.35	0.36	0.36
100	75	0.75	1.01	0.59	0.35	0.43	0.48	0.52	0.54	0.58
100	75	0.85	0.99	0.60	0.38	0.48	0.57	0.64	0.68	0.77
100	75	0.95	1.00	0.63	0.43	0.56	0.68	0.76	0.81	0.94
100	75	1.00	1.00	0.65	0.45	0.59	0.72	0.81	0.87	1.01
100	100	0.50	0.97	0.56	0.30	0.31	0.33	0.33	0.33	0.34
100	100	0.75	0.98	0.56	0.33	0.40	0.45	0.48	0.50	0.52
100	100	0.85	0.99	0.57	0.36	0.45	0.54	0.60	0.63	0.69
100	100	0.95	0.99	0.62	0.41	0.54	0.66	0.74	0.79	0.91
100	100	1.00	0.99	0.63	0.43	0.56	0.68	0.77	0.83	0.97
200	200	0.50	0.91	0.36	0.14	0.15	0.16	0.16	0.16	0.16
200	200	0.75	0.92	0.39	0.16	0.19	0.22	0.24	0.26	0.30
200	200	0.85	0.93	0.42	0.19	0.22	0.27	0.32	0.35	0.49
200	200	0.95	0.93	0.45	0.23	0.28	0.36	0.43	0.49	0.77
200	200	1.00	0.95	0.48	0.25	0.30	0.39	0.47	0.55	0.90
300	300	0.50	0.87	0.28	0.10	0.10	0.11	0.11	0.11	0.11
300	300	0.75	0.88	0.30	0.11	0.13	0.15	0.16	0.18	0.22
300	300	0.85	0.89	0.32	0.13	0.15	0.19	0.22	0.25	0.41
300	300	0.95	0.90	0.37	0.16	0.18	0.23	0.28	0.33	0.68
300	300	1.00	0.90	0.41	0.19	0.21	0.26	0.31	0.37	0.77

Ratio between the MSE for the estimated IRFs obtained by fitting a VECM on $\widehat{\mathbf{F}}_t$, and the MSE for the estimated and cumulated IRFs obtained by estimating a VAR on $\Delta\widehat{\mathbf{F}}_t$ as in Forni et al. (2009). Values smaller than one indicate a better performance of our method. T is the number of observations, n is the number of variables, and $n_1 = \lceil n^\delta \rceil$ is the number of $I(1)$ idiosyncratic components. In these simulations there are $n_b = \lceil n^\eta \rceil$ variables with a deterministic linear trend, with $\eta = \delta$ or equivalently $n_b = n_1$.

Table F14a: MonteCarlo Simulations - Impulse Responses
Mean Squared Errors relative to VAR in Differences
All variables, All Shocks

T	n	δ	η	$k=0$	$k=1$	$k=4$	$k=8$	$k=12$	$k=16$	$k=20$	$k=100$
75	50	0.50	0.50	1.18	0.81	0.58	0.63	0.65	0.67	0.68	0.69
75	50	0.75	0.50	1.47	1.16	0.94	0.93	0.97	1.01	1.04	1.11
75	50	0.75	0.75	1.10	0.75	0.59	0.68	0.74	0.77	0.79	0.81
75	50	0.85	0.50	1.26	0.86	0.64	0.72	0.80	0.83	0.86	0.91
75	50	0.85	0.75	1.23	0.89	0.64	0.70	0.79	0.85	0.86	0.89
75	50	0.85	0.85	1.24	0.86	0.69	0.78	0.86	0.90	0.92	0.96
75	50	0.95	0.50	1.23	0.87	0.67	0.78	0.88	0.93	0.97	1.05
75	50	0.95	0.75	0.98	0.81	0.66	0.76	0.84	0.89	0.92	0.96
75	50	0.95	0.85	1.27	0.90	0.69	0.79	0.86	0.91	0.95	1.00
75	50	0.95	0.95	1.33	0.93	0.75	0.84	0.89	0.92	0.94	0.97
75	50	1.00	0.50	1.14	0.83	0.69	0.79	0.88	0.95	0.98	1.06
75	50	1.00	0.75	1.23	0.93	0.77	0.87	0.97	1.04	1.08	1.16
75	50	1.00	0.85	1.15	0.86	0.67	0.79	0.88	0.93	0.95	0.98
75	50	1.00	0.95	1.21	0.90	0.77	0.88	0.96	1.00	1.01	1.04
75	50	1.00	1.00	0.94	0.68	0.59	0.71	0.79	0.82	0.83	0.85
100	50	0.50	0.50	1.07	0.66	0.41	0.44	0.47	0.49	0.50	0.52
100	50	0.75	0.50	0.96	0.65	0.51	0.55	0.60	0.63	0.66	0.72
100	50	0.75	0.75	0.93	0.67	0.49	0.55	0.61	0.65	0.67	0.71
100	50	0.85	0.50	1.13	0.76	0.54	0.62	0.71	0.76	0.80	0.89
100	50	0.85	0.75	1.02	0.70	0.52	0.63	0.73	0.80	0.84	0.92
100	50	0.85	0.85	1.23	0.78	0.54	0.64	0.72	0.77	0.81	0.87
100	50	0.95	0.50	1.08	0.76	0.57	0.69	0.80	0.88	0.93	1.05
100	50	0.95	0.75	1.18	0.82	0.59	0.69	0.80	0.87	0.92	1.02
100	50	0.95	0.85	1.08	0.76	0.57	0.70	0.81	0.89	0.94	1.03
100	50	0.95	0.95	0.94	0.71	0.55	0.65	0.75	0.81	0.85	0.91
100	50	1.00	0.50	1.18	0.84	0.61	0.71	0.83	0.91	0.97	1.08
100	50	1.00	0.75	1.03	0.76	0.57	0.70	0.82	0.90	0.95	1.05
100	50	1.00	0.85	1.13	0.83	0.63	0.74	0.84	0.91	0.95	1.04
100	50	1.00	0.95	1.15	0.82	0.59	0.70	0.82	0.89	0.93	1.00
100	50	1.00	1.00	1.07	0.77	0.61	0.72	0.83	0.89	0.93	0.99

Ratio between the MSE for the estimated IRFs obtained by fitting a VECM on $\widehat{\mathbf{F}}_t$, and the MSE for the estimated and cumulated IRFs obtained by estimating a VAR on $\Delta\widetilde{\mathbf{F}}_t$ as in Forni et al. (2009). Values smaller than one indicate a better performance of our method. T is the number of observations, n is the number of variables, and $n_1 = \lceil n^\delta \rceil$ is the number of $I(1)$ idiosyncratic components, and $n_b = \lceil n^\eta \rceil$ is the number of variables with a deterministic linear trend.

Table F14b: MonteCarlo Simulations - Impulse Responses
Mean Squared Errors relative to VAR in Differences
All variables, All Shocks

T	n	δ	η	$k=0$	$k=1$	$k=4$	$k=8$	$k=12$	$k=16$	$k=20$	$k=100$
75	75	0.50	0.50	1.00	0.69	0.50	0.53	0.54	0.54	0.54	0.55
75	75	0.75	0.50	1.01	0.76	0.61	0.63	0.65	0.68	0.69	0.71
75	75	0.75	0.75	1.06	0.81	0.60	0.66	0.71	0.73	0.75	0.77
75	75	0.85	0.50	1.13	0.81	0.59	0.63	0.68	0.72	0.75	0.80
75	75	0.85	0.75	1.08	0.77	0.61	0.66	0.72	0.76	0.77	0.81
75	75	0.85	0.85	1.06	0.73	0.59	0.68	0.76	0.80	0.81	0.84
75	75	0.95	0.50	1.07	0.80	0.60	0.70	0.81	0.87	0.90	0.97
75	75	0.95	0.75	1.19	0.84	0.63	0.71	0.79	0.84	0.87	0.91
75	75	0.95	0.85	1.14	0.75	0.60	0.74	0.85	0.91	0.94	0.98
75	75	0.95	0.95	1.02	0.73	0.60	0.75	0.85	0.89	0.90	0.92
75	75	1.00	0.50	1.11	0.81	0.63	0.75	0.86	0.93	0.97	1.04
75	75	1.00	0.75	1.30	0.89	0.66	0.78	0.89	0.95	0.98	1.02
75	75	1.00	0.85	1.07	0.77	0.63	0.76	0.87	0.93	0.96	0.99
75	75	1.00	0.95	1.03	0.72	0.59	0.75	0.86	0.90	0.92	0.94
75	75	1.00	1.00	0.92	0.71	0.57	0.69	0.77	0.80	0.81	0.83
100	75	0.50	0.50	0.95	0.60	0.36	0.38	0.39	0.40	0.41	0.41
100	75	0.75	0.50	1.01	0.65	0.43	0.49	0.54	0.57	0.60	0.63
100	75	0.75	0.75	1.03	0.69	0.44	0.49	0.54	0.57	0.59	0.62
100	75	0.85	0.50	1.05	0.67	0.44	0.53	0.61	0.67	0.70	0.78
100	75	0.85	0.75	1.02	0.67	0.46	0.54	0.61	0.65	0.68	0.73
100	75	0.85	0.85	1.04	0.68	0.45	0.55	0.63	0.69	0.72	0.76
100	75	0.95	0.50	1.03	0.72	0.51	0.61	0.71	0.78	0.83	0.93
100	75	0.95	0.75	1.64	1.06	0.64	0.69	0.76	0.82	0.86	0.93
100	75	0.95	0.85	1.05	0.72	0.52	0.64	0.76	0.83	0.88	0.97
100	75	0.95	0.95	1.04	0.68	0.49	0.63	0.74	0.81	0.85	0.92
100	75	1.00	0.50	1.09	0.74	0.53	0.64	0.75	0.83	0.88	0.99
100	75	1.00	0.75	1.10	0.76	0.57	0.65	0.75	0.82	0.87	0.98
100	75	1.00	0.85	0.98	0.69	0.52	0.65	0.76	0.84	0.89	0.98
100	75	1.00	0.95	1.10	0.72	0.52	0.66	0.77	0.84	0.88	0.94
100	75	1.00	1.00	1.03	0.70	0.52	0.67	0.79	0.87	0.91	0.97

Ratio between the MSE for the estimated IRFs obtained by fitting a VECM on $\widehat{\mathbf{F}}_t$, and the MSE for the estimated and cumulated IRFs obtained by estimating a VAR on $\Delta\widetilde{\mathbf{F}}_t$ as in Forni et al. (2009). Values smaller than one indicate a better performance of our method. T is the number of observations, n is the number of variables, and $n_1 = \lceil n^{\delta} \rceil$ is the number of $I(1)$ idiosyncratic components, and $n_b = \lceil n^{\eta} \rceil$ is the number of variables with a deterministic linear trend.

Table F14c: MonteCarlo Simulations - Impulse Responses
Mean Squared Errors relative to VAR in Differences
All variables, All Shocks

T	n	δ	η	$k=0$	$k=1$	$k=4$	$k=8$	$k=12$	$k=16$	$k=20$	$k=100$
100	100	0.50	0.50	0.99	0.62	0.36	0.37	0.38	0.38	0.38	0.39
100	100	0.75	0.50	1.02	0.63	0.39	0.45	0.50	0.53	0.54	0.57
100	100	0.75	0.75	0.96	0.61	0.40	0.45	0.49	0.52	0.54	0.56
100	100	0.85	0.50	1.13	0.74	0.50	0.56	0.63	0.68	0.72	0.79
100	100	0.85	0.75	1.00	0.66	0.45	0.53	0.60	0.65	0.68	0.74
100	100	0.85	0.85	1.01	0.66	0.45	0.54	0.61	0.66	0.69	0.75
100	100	0.95	0.50	1.02	0.66	0.45	0.57	0.68	0.76	0.82	0.93
100	100	0.95	0.75	1.01	0.65	0.45	0.57	0.68	0.76	0.80	0.90
100	100	0.95	0.85	1.04	0.69	0.48	0.59	0.71	0.78	0.83	0.91
100	100	0.95	0.95	1.02	0.66	0.45	0.58	0.70	0.77	0.81	0.87
100	100	1.00	0.50	1.01	0.67	0.47	0.61	0.72	0.80	0.85	0.95
100	100	1.00	0.75	1.04	0.66	0.47	0.61	0.74	0.83	0.89	1.00
100	100	1.00	0.85	0.99	0.65	0.46	0.60	0.72	0.80	0.85	0.95
100	100	1.00	0.95	0.99	0.65	0.48	0.64	0.78	0.85	0.89	0.95
100	100	1.00	1.00	1.02	0.68	0.47	0.61	0.72	0.78	0.82	0.86
200	200	0.50	0.50	0.94	0.43	0.19	0.19	0.20	0.20	0.20	0.20
200	200	0.75	0.50	0.94	0.43	0.19	0.22	0.25	0.27	0.29	0.34
200	200	0.75	0.75	0.94	0.45	0.20	0.22	0.24	0.26	0.28	0.32
200	200	0.85	0.50	0.94	0.45	0.22	0.26	0.31	0.35	0.39	0.52
200	200	0.85	0.75	0.95	0.44	0.22	0.26	0.31	0.36	0.39	0.52
200	200	0.85	0.85	0.95	0.45	0.21	0.25	0.30	0.35	0.39	0.51
200	200	0.95	0.50	0.95	0.49	0.25	0.30	0.37	0.44	0.50	0.77
200	200	0.95	0.75	0.96	0.49	0.24	0.30	0.38	0.46	0.52	0.79
200	200	0.95	0.85	0.96	0.49	0.25	0.31	0.40	0.48	0.55	0.81
200	200	0.95	0.95	0.95	0.48	0.24	0.31	0.40	0.48	0.54	0.78
200	200	1.00	0.50	0.96	0.51	0.26	0.32	0.40	0.48	0.55	0.87
200	200	1.00	0.75	0.95	0.50	0.26	0.32	0.40	0.48	0.56	0.86
200	200	1.00	0.85	0.96	0.52	0.27	0.34	0.44	0.53	0.61	0.92
200	200	1.00	0.95	0.95	0.50	0.27	0.35	0.44	0.53	0.61	0.88
200	200	1.00	1.00	0.95	0.48	0.26	0.34	0.44	0.52	0.59	0.84
300	300	0.50	0.50	0.90	0.32	0.12	0.12	0.13	0.13	0.13	0.13
300	300	0.75	0.50	0.91	0.33	0.13	0.14	0.16	0.17	0.18	0.23
300	300	0.75	0.75	0.91	0.33	0.13	0.14	0.15	0.17	0.18	0.23
300	300	0.85	0.50	0.92	0.36	0.15	0.17	0.20	0.23	0.26	0.43
300	300	0.85	0.75	0.91	0.36	0.15	0.17	0.20	0.23	0.26	0.41
300	300	0.85	0.85	0.91	0.36	0.15	0.17	0.20	0.23	0.25	0.40
300	300	0.95	0.50	0.93	0.41	0.18	0.20	0.25	0.30	0.36	0.74
300	300	0.95	0.75	0.93	0.40	0.17	0.20	0.25	0.30	0.36	0.71
300	300	0.95	0.85	0.92	0.40	0.17	0.20	0.26	0.32	0.37	0.73
300	300	0.95	0.95	0.91	0.39	0.18	0.21	0.27	0.33	0.39	0.73
300	300	1.00	0.50	0.92	0.42	0.19	0.22	0.27	0.33	0.39	0.83
300	300	1.00	0.75	0.93	0.43	0.19	0.22	0.27	0.33	0.38	0.79
300	300	1.00	0.85	0.92	0.42	0.19	0.22	0.28	0.34	0.40	0.80
300	300	1.00	0.95	0.92	0.41	0.19	0.22	0.29	0.36	0.43	0.84
300	300	1.00	1.00	0.92	0.40	0.18	0.22	0.28	0.35	0.41	0.79

Ratio between the MSE for the estimated IRFs obtained by fitting a VECM on $\widehat{\mathbf{F}}_t$, and the MSE for the estimated and cumulated IRFs obtained by estimating a VAR on $\Delta\widetilde{\mathbf{F}}_t$ as in Forni et al. (2009). Values smaller than one indicate a better performance of our method. T is the number of observations, n is the number of variables, and $n_1 = \lceil n^{\delta} \rceil$ is the number of $I(1)$ idiosyncratic components, and $n_b = \lceil n^{\eta} \rceil$ is the number of variables with a deterministic linear trend.

Table F15: MonteCarlo Simulations - Number of Common Trends and Shocks
Percentage of Correct Answer

T	n	δ	η	$\widehat{\tau} = \tau$	$\widehat{q} = q$	T	n	δ	η	$\widehat{\tau} = \tau$	$\widehat{q} = q$
75	50	0.50	0.50	91.5	49.7	100	75	0.50	0.50	89.1	86.5
75	50	0.75	0.50	96.2	49.9	100	75	0.75	0.50	97.3	87.8
75	50	0.75	0.75	97.3	52.7	100	75	0.75	0.75	98.2	87.3
75	50	0.85	0.50	98.1	54.7	100	75	0.85	0.50	99.1	86.4
75	50	0.85	0.75	97.3	53.5	100	75	0.85	0.75	99.4	86.8
75	50	0.85	0.85	97.8	55.1	100	75	0.85	0.85	99.3	86.7
75	50	0.95	0.50	95.8	66.7	100	75	0.95	0.50	99.3	91.3
75	50	0.95	0.75	96.0	64.6	100	75	0.95	0.75	99.1	90.8
75	50	0.95	0.85	96.2	65.1	100	75	0.95	0.85	99.0	91.9
75	50	0.95	0.95	96.4	64.7	100	75	0.95	0.95	99.0	92.4
75	50	1.00	0.50	95.5	73.6	100	75	1.00	0.50	98.5	95.9
75	50	1.00	0.75	95.2	73.3	100	75	1.00	0.75	99.1	95.3
75	50	1.00	0.85	94.2	72.9	100	75	1.00	0.85	98.4	95.7
75	50	1.00	0.95	94.6	72.5	100	75	1.00	0.95	98.3	96.4
75	50	1.00	1.00	95.3	74.1	100	75	1.00	1.00	98.2	95.8
100	50	0.50	0.50	93.3	60.6	100	100	0.50	0.50	82.2	96.4
100	50	0.75	0.50	97.7	59.7	100	100	0.75	0.50	96.5	95.9
100	50	0.75	0.75	98.4	61.0	100	100	0.75	0.75	96.6	95.9
100	50	0.85	0.50	98.3	63.5	100	100	0.85	0.50	99.2	97.2
100	50	0.85	0.75	98.1	64.3	100	100	0.85	0.75	99.3	96.6
100	50	0.85	0.85	98.1	64.2	100	100	0.85	0.85	99.3	95.9
100	50	0.95	0.50	97.7	72.5	100	100	0.95	0.50	99.6	98.3
100	50	0.95	0.75	97.1	74.5	100	100	0.95	0.75	99.6	97.9
100	50	0.95	0.85	97.8	74.3	100	100	0.95	0.85	99.3	97.5
100	50	0.95	0.95	97.0	71.1	100	100	0.95	0.95	99.5	98.0
100	50	1.00	0.50	95.9	82.4	100	100	1.00	0.50	99.2	99.3
100	50	1.00	0.75	96.6	82.5	100	100	1.00	0.75	99.3	99.4
100	50	1.00	0.85	95.5	83.8	100	100	1.00	0.85	99.5	99.2
100	50	1.00	0.95	96.1	82.9	100	100	1.00	0.95	99.2	99.1
100	50	1.00	1.00	96.3	84.4	100	100	1.00	1.00	99.0	99.3
75	75	0.50	0.50	84.9	80.0	200	200	0.50	0.50	70.5	100.0
75	75	0.75	0.50	96.6	79.1	200	200	0.75	0.50	94.1	100.0
75	75	0.75	0.75	97.1	78.8	200	200	0.75	0.75	93.0	100.0
75	75	0.85	0.50	98.8	81.1	200	200	0.85	0.50	98.7	100.0
75	75	0.85	0.75	97.9	79.9	200	200	0.85	0.75	99.0	100.0
75	75	0.85	0.85	98.9	79.7	200	200	0.85	0.85	98.5	100.0
75	75	0.95	0.50	98.0	85.1	200	200	0.95	0.50	100.0	100.0
75	75	0.95	0.75	98.6	84.3	200	200	0.95	0.75	100.0	100.0
75	75	0.95	0.85	98.1	84.0	200	200	0.95	0.85	99.9	100.0
75	75	0.95	0.95	98.4	85.3	200	200	0.95	0.95	99.9	100.0
75	75	1.00	0.50	97.6	90.5	200	200	1.00	0.50	100.0	100.0
75	75	1.00	0.75	98.0	91.7	200	200	1.00	0.75	100.0	100.0
75	75	1.00	0.85	98.0	91.0	200	200	1.00	0.85	100.0	100.0
75	75	1.00	0.95	97.7	91.7	200	200	1.00	0.95	100.0	100.0
75	75	1.00	1.00	97.3	91.3	200	200	1.00	1.00	100.0	100.0

Percentage of cases in which we estimate the correct number of all common shocks ($\widehat{q} = q$) and of common permanent shocks ($\widehat{\tau} = \tau$). T is the number of observations, n is the number of variables, and $n_1 = \lceil n^{\delta} \rceil$ is the number of $I(1)$ idiosyncratic components, and $n_b = \lceil n^{\eta} \rceil$ is the number of variables with a deterministic linear trend.