Nonfundamentalness in Structural Econometric Models: A Review.

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Abstract

Economic theory typically assumes the existence of few unobserved unpredictable stochastic disturbances, called *structural shocks*, driving the whole economy. Would the economy be representable as a very high dimensional stochastic vector process, those shocks would be the reduced rank *innovation* of that process. In practice, however, only a few components of that process are observable, the innovation of which, in general, does not coincide with the structural shocks. As a result, a MA representation of the observed process in terms of the structural shocks will be a noninvertible one. In other words, the structural shocks driving the economy do not belong to the past of the observed series, but also involve their future, i.e. they are *nonfundamental* for the observed process. It follows that the present values of those structural shocks cannot be recovered from the observations; in particular, fitting causal VARs to the observed series can be extremely misleading. We review economic literature on VARs and we provide many examples of small size economic models that imply nonfundamentalness. If fundamental shocks nevertheless are to be recovered, the only solution consists in enlarging the space of observations, that is, in considering a very *large panel* of related time series. Among several alternatives, we consider the dynamic factor model methodology, which requires very little additional assumptions. We review first their economic interpretation and then we show how fundamentalness of the shocks is always guaranteed in this framework. Finally, by means of structural dynamic factor models we provide new empirical evidence on the relation between hours worked and technology shocks and we compare it with traditional VAR results.

Keywords: Nonfundamentalness, Noninvertible MA, Structural VAR, Dynamic Stochastic General Equilibrium Models, Dynamic Factor Models.

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Contents

1 Introduction

Autoregressive moving average, or ARMA, processes are of fundamental importance when analyzing time series. Any weakly stationary time series with exponentially decaying autocovariance function can be approximated, with arbitrary precision, as an ARMA with adequate lags. Usually, we assume that the AR component in an ARMA model is causal and that the MA component is invertible. These conditions in the univariate case reduce to the requirement that the complex roots of the two polynomials lie all outside the unit circle; in a multivariate setting, the same conditions are required for the roots of the determinants of the AR and MA polynomial matrices. In this paper we investigate the consequences of considering a noninvertible MA component in econometrics.

We assume that all the stochastic variables considered in this paper belong to the Hilbert space $L^2(\Omega, \mathcal{F}, P)$ for some given probability space $(\Omega, \mathcal{F}, P)$. Let us consider an N-dimensional, covariance stationary, zero-mean vector stochastic process $\{\mathbf{x}_t\,, t \in \mathbb{Z}\}$ with rational spectral density.[1] We define the Hilbert space generated by all the observations of $\mathbf{x}_t$ up to time t as $\mathcal{H}_t^x \subset L^2(\Omega, \mathcal{F}, P)$, i.e. $\mathcal{H}_t^x = \overline{\text{span}}\,\{\mathbf{x}_{t-k}, k \geq 0\}$. The process $\mathbf{x}_t$ is a VARMA(p_1, p_2) process if it is a stationary solution of

$$\mathbf{\Phi}(L)\mathbf{x}_t = \mathbf{\Theta}(L)\mathbf{u}_t, \quad \mathbf{u}_t \sim \text{w.n.}(\mathbf{0}, \mathbf{\Gamma}_0^u), \quad t \in \mathbb{Z}. \tag{1}$$

The AR and MA polynomial matrices are respectively defined as $\mathbf{\Phi}(L) = \mathbf{I} - \sum_{k=1}^{p_1} \mathbf{\Phi}_k L^k$ and $\mathbf{\Theta}(L) = \mathbf{I} + \sum_{k=1}^{p_2} \mathbf{\Theta}_k L^k$, where $\mathbf{I}$ is the N-dimensional identity matrix and L is the lag-operator. The vector $\mathbf{u}_t$ is an N-dimensional weak white noise process with covariance matrix $\mathbf{\Gamma}_0^u$.[2] The process $\mathbf{x}_t$ always admits the Wold decompostion

$$\mathbf{x}_t = \boldsymbol{\varepsilon}_t^x + \sum_{k=1}^{\infty} \mathbf{\Psi}_k \boldsymbol{\varepsilon}_{t-k}^x + \boldsymbol{\delta}_t, \quad \boldsymbol{\varepsilon}_t^x \sim \text{w.n.}(\mathbf{0}, \mathbf{\Gamma}_0^{\varepsilon^x}), \quad t \in \mathbb{Z}, \tag{2}$$

where $\boldsymbol{\delta}_t$ is a purely deterministic process. For simplicity in this paper we treat only *purely undeterministic* processes, therefore $\boldsymbol{\delta}_t = \mathbf{0}$. In (2) $\boldsymbol{\varepsilon}_t^x$ is the innovation process of $\mathbf{x}_t$ and, by definition, it belongs to the space generated by $\mathbf{x}_t$. Moreover, since we are in a purely undeterministic case, the space generated by $\mathbf{x}_t$ and the space generated by $\boldsymbol{\varepsilon}_t^x$ coincide at any t, i.e. $\mathcal{H}_t^{\varepsilon^x} = \mathcal{H}_t^x$. From (1), we see that $\mathbf{x}_t$ is also associated to a white noise process $\mathbf{u}_t$. However, $\mathbf{u}_t$ does not necessarily generate the same space as the space generated by the innovations process $\boldsymbol{\varepsilon}_t^x$. Indeed, there are two possibilities. If (1) is assumed to be causal and invertible, then $\mathcal{H}_t^u = \mathcal{H}_t^{\varepsilon^x} = \mathcal{H}_t^x$ for any t, and we say that $\mathbf{u}_t$ is $\mathbf{x}_t$-*fundamental.* If instead we assume (1) to be causal but noninvertible, then, at any t, $\mathcal{H}_t^{\varepsilon^x} \subset \mathcal{H}_t^u$, i.e. $\mathcal{H}_t^x \subset \mathcal{H}_t^u$, and we say that $\mathbf{u}_t$ is $\mathbf{x}_t$-*nonfundamental.* Although in both cases (1) generates the same autocovariance structure, a $\mathbf{x}_t$-nonfundamental and a $\mathbf{x}_t$-fundamental white noise may have very different implications for economic theory. While keeping the assumption of a causal VARMA, in this paper we investigate the consequences of a noninvertible VARMA when validating economic models by methods of time series analysis.

Dynamic Stochastic General Equilibrium (DSGE) models are often used in Central Banks for modeling and analyzing national and worldwide economies. These are analytical models based on finite difference equations with stochastic disturbances. They are used for aggregating the unobserved behavior of single agents (e.g. consumers or firms) in observed macroeconomic time series. By log-linearization around their steady-state, DSGE models can be reduced to VARMA models. Let us call $\mathbf{y}_t$ a large, possibly infinte, dimensional vector process containing all available economic information. This process is in general not observable, but still we know that at each t it generates a space which coincides with the space of its innovations $\boldsymbol{\varepsilon}_t^y$. Moreover, we know that a VARMA representation of the DSGE model for $\mathbf{y}_t$ has an associated white noise $\mathbf{u}_t$. A reasonable assumption underlying economic models is that $\mathcal{H}_t^u = \mathcal{H}_t^{\varepsilon^y} = \mathcal{H}_t^y$, for any t. Indeed, econometricians interpret $\mathbf{u}_t$ as a vector of unexpected shocks affecting the economy as a whole and, for this reason, $\mathbf{u}_t$ is always assumed as $\mathbf{y}_t$-fundamental, i.e. the economy-wide VARMA representation of the DSGE model must have an invertible MA component. In order to emphasize their economic meaning, we call $\mathbf{u}_t$ *structural shocks.* Typically

[1] Hereafter we indicate any stochastic process $\{\mathbf{x}_t\,, t \in \mathbb{Z}\}$ simply as $\mathbf{x}_t$.

[2] Hereafter for simplicity we indicate any *weak* white noise processs as white noise or even simply as shocks.

these shocks are very few (i.e. they have reduced rank covariance matrix), and can be, for example, techological (e.g. a new technical innovation), monetary (e.g. a change in the interest rate), fiscal (e.g. an increase in taxes or in government spending), financial (e.g. a crash or a bubble in the stock market).

Usually, econometricians observe only a finite and often small amount of information, i.e. they observe only a subset $\mathbf{x}_t$ of the economy wide process $\mathbf{y}_t$. By definition, the structural shocks $\mathbf{u}_t$ that affect the whole economy $\mathbf{y}_t$ affect also a subset $\mathbf{x}_t$ of it. However, in this paper we provide evidence of many economic examples for which the $\mathbf{y}_t$-fundamental process $\mathbf{u}_t$ becomes $\mathbf{x}_t$-nonfundamental when considering small subsets $\mathbf{x}_t$ of the economy-wide process $\mathbf{y}_t$. Therefore, the space generated by $\mathbf{u}_t$ coincides with the space generated by the innovations of $\mathbf{y}_t$, but it is larger than the space generated by the innovations of $\mathbf{x}_t$:

$$\begin{array}{ccccc} \mathcal{H}_t^x & \subset & \mathcal{H}_t^u & = & \mathcal{H}_t^y \\ \| & & & & \| \\ \mathcal{H}_t^{\varepsilon^x} & \subset & \mathcal{H}_t^u & = & \mathcal{H}_t^{\varepsilon^y} \end{array}, \qquad t \in \mathbb{Z}. \tag{3}$$

In order to be able to retrieve the space generated by the structural shocks $\mathbf{u}_t$ from a small size panel $\mathbf{x}_t$, we would need to assume a noninvertible VARMA model. If (1) is noninvertible then $\mathbf{u}_t$ belongs to the space spanned by future values of $\mathbf{x}_t$. This feature is clearly not credible in any real world model. What is then the meaning of nonfundamental shocks? From (3) we see that $\mathbf{u}_t$ is $\mathbf{x}_t$-nonfundamental because the econometricians' information space is smaller than the information space of an agent able to observe the whole economy $\mathbf{y}_t$. In this sense nonfundamentalness is a problem related to information.

The traditional way to retrieve the space generated by the structural shocks $\mathcal{H}_t^u$ consists in assuming finite lag, causal, VAR approximations of a VARMAs for a small size process $\mathbf{x}_t$. By definition of causal VAR, the associated white noise, say $\mathbf{v}_t$ is $\mathbf{x}_t$-fundamental. Thus, a VAR approximation is valid only as long as the structural shocks $\mathbf{u}_t$ are $\mathbf{x}_t$-fundamental. If instead economic theory tells us that $\mathbf{u}_t$ is $\mathbf{x}_t$-nonfundamental, then VARs are not the right model to use since in this case $\mathcal{H}_t^v \subset \mathcal{H}_t^u$, for any t. The examples of nonfundamentalness we provide in this paper arise in small size economic models which are precisely the kind of models usually studied by means of VARs. In this sense we can consider the present work as a critique of VARs as tools for validating economic theories.

Two solutions are possible to solve the problem of nonfundamentalness. Either we apply to $\mathbf{v}_t$ *ad hoc* two-sided filters called Blaschke matrices in order to obtain a new white noise process $\widetilde{\mathbf{v}}_t$ such that $\mathcal{H}_t^{\tilde{v}} = \mathcal{H}_t^u$, for any t. Or we enlarge the econometricians' information space in such a way that $\mathcal{H}_t^x = \mathcal{H}_t^y$, for any t. This second way is based mainly on factor models, as those proposed in Forni et al. (2000) and Forni et al. (2009), which are nowdays a popular class of models apt to deal with very large datasets mimicking the economy-wide process $\mathbf{y}_t$.

In this paper, we summarize and organize existing results on nonfundamentalness in the econometric literature with a twofold objective. First, we would like to convince the reader that there are many meaningful econometric models which can be generated by nonfundamental shocks, thus the existence of the problem should not be a secondary aspect in econometric modeling. Second, we want to stimulate the debate in the econometrics and statistics communities which, although both aware of the problem, often neglect it. Indeed, while econometricians tend to stick to traditional VARs techiniques (e.g. supplying the missing information just by adding more and more lags to VARs), on the other hand statisticians are often not aware of the potential harmful effects of neglecting nonfundamentalness when validating economic models by means of VARs.

In the next Section, we give necessary and sufficient conditions for fundamentalness in full- and reduced-rank systems. In Section 3 we briefly review the estimation and identification of the structural shocks in VARs. In Section 4 we illustrate the debate between Blanchard and Quah (1989) *vs.* Lippi and Reichlin (1993) as a textbook example of how a simple bidimensional economically meaningful model can be generated by a nonfundamental white noise. In Section 5 we start from the works by Hansen and Sargent (1980, 1991) and we look at nonfundamentalness in rational expectations models. Once explained why nonfundamental representations cannot be ignored, in Section 6 we review consider fundamentalness when validating DSGE models by means of VARs. While in Section 7 we show the value of adding new information by following the argument by Giannone and Reichlin (2006) based on Granger causality. Section 8 is based on Forni et al. (2009) and factors models are proposed as an alternative tool for identifying structural shocks. First we introduce the model as a representation of DSGE models with measurement errors, we show how to deal with nonfundamentalness in this case and finally we show how to estimate and identify the structural shocks. In Section 9 we add a new piece of evidence on nonfundamentalness by using use factor models to retrieve the impulse response of hours worked to a technology shock and we compare our results with those obtained with a bivariate VAR by Christiano et al. (2004). In Section 10 we conclude and suggest developments for future research.

2 Conditions for Fundamentalness

Hereafter for simplicty we consider just multivariate MA processes instead of VARMA. Indeed, once causality is assumed, our focus is only on the MA component and no other condition is required for the AR component. Consider an N-dimensional MA process $\mathbf{x}_t$

$$\mathbf{x}_t = \mathbf{C}(L)\mathbf{u}_t, \quad \mathbf{u}_t \sim \text{w.n.}(\mathbf{0}, \mathbf{\Gamma}_0^u), \quad t \in \mathbb{Z}, \tag{4}$$

where $\mathbf{u}_t$ is a q-dimensional vector white noise process of structural shocks with $q \neq N$ and $\mathbf{C}(L)$ is a onesided $N \times q$ matrix polynomial in the lag operator L. In principle $\mathbf{C}(L)$ can be of infinite order with square-summable coefficients, i.e. $\mathbf{C}(L) = \sum_{k=0}^{\infty} \mathbf{C}_k L^k$ such that $\sum_{k=0}^{\infty} |(\mathbf{C}_k)_{i,j}|^2 < \infty$ for any $i, j = 1, \dots N$.

Definition (Fundamentalness) *We say that* $\mathbf{u}_t$ *is* $\mathbf{x}_t$*-fundamental if* $\mathcal{H}_t^x = \mathcal{H}_t^u$*, for any* $t \in \mathbb{Z}$*. Whereas, we say that* $\mathbf{u}_t$ *is* $\mathbf{x}_t$*-nonfundamental if* $\mathcal{H}_t^x \subset \mathcal{H}_t^u$*, for any* $t \in \mathbb{Z}$*.*

If $q > N$ then $\mathbf{u}_t$ is $\mathbf{x}_t$-nonfundamental as we observe less variables than shocks. Therefore, a necessary condition for fundamentalness is $q \leq N$. Let us start by considering the case $N = q$, then the necessary and sufficient condition for fundamentalness of $\mathbf{u}_t$ is the following (see e.g. Rozanov, 1967):

Condition 1 *Given a covariance stationary vector process* $\mathbf{x}_t$*, with rational spectral density, with MA representation* $\mathbf{x}_t = \mathbf{C}(L)\mathbf{u}_t$*, the process* $\mathbf{u}_t$ *is fundamental for* $\mathbf{x}_t$ *if and only if*

1. $\mathbf{u}_t$ *is a weak white noise vector;*

2. $\mathbf{C}(z)$ *has no poles inside the unit circle;*

3. $\det \mathbf{C}(z)$ *has all its roots lying outside the unit circle:*

$$\det \mathbf{C}(z) \neq 0 \quad \forall\, z \in \mathbb{C} \quad \text{s.t. } |z| < 1\,.$$

On the other hand, in the singular case $N > q$, we have the following necessary and sufficient condition for fundamentalness of $\mathbf{u}_t$ (see e.g. Rozanov, 1967):

Condition 2 *Given a covariance stationary vector process* $\mathbf{x}_t$*, with rational spectral density, with MA representation* $\mathbf{x}_t = \mathbf{C}(L)\mathbf{u}_t$*, where* $\mathbf{C}(L)$ *is* $N \times q$ *and* $N > q$*, the process* $\mathbf{u}_t$ *is fundamental for* $\mathbf{x}_t$ *if and only if:*

1. $\mathbf{u}_t$ *is a weak white noise vector;*

2. $\mathbf{C}(z)$ *has no poles inside the unit circle;*

3. $\mathbf{C}(z)$ *has full rank inside the unit circle*

$$\text{rank } \mathbf{C}(z) = q \quad \forall\, z \in \mathbb{C} \quad \text{s.t. } |z| < 1\,.$$

We can restate this last condition in terms of the roots of $\det \mathbf{C}(z)$*. We ask the determinants of all the* $q \times q$ *submatrices of* $\mathbf{C}(z)$ *to have no common roots inside the unit circle. More precisely, if we call* $\mathbf{C}_j(z)$ *the submatrices contained in* $\mathbf{C}(z)$ *and we define the set of indexes* $\mathbb{I} = \left\{ j \in \mathbb{N} \text{ s.t. } j = 1, \ldots, \binom{N}{q} \right\}$*, the condition for fundamentalness requires that*

$$\nexists\, z \in \mathbb{C} \;\text{ s.t. }\; |z| < 1 \quad \textit{and} \quad \det \mathbf{C}_j(z) = 0 \quad \forall\, j \in \mathbb{I}\,.$$

Remark 1 (Invertibility and fundamentalness) We say that $\mathbf{C}(z)$ is invertible if it exists a $q \times N$ full-rank polynomial matrix $\mathbf{D}(z)$ such that $\mathbf{D}(z)'\mathbf{C}(z) = \mathbf{I}$. Both previous conditions imply that $\mathbf{D}(z)$ exists and is composed only of positive powers of z, thus present and past values of $\mathbf{x}_t$ are enough to recover $\mathbf{u}_t$. This is another way of defining fundamentalness. On the other hand, if conditions 1 or 2 do not hold, we have two possibilities. Either $\mathbf{D}(z)$ still exists but is made also of negative powers of z, so that we need also future values of $\mathbf{x}_t$ to recover $\mathbf{u}_t$ and thus $\mathbf{u}_t$ is $\mathbf{x}_t$-nonfundamental. Or conditions 1 and 2 are violated for some $z \in \mathbb{C}$ such that $|z| = 1$, so that $\mathbf{D}(z)$ does not exist. In this case we could prove that $\mathbf{u}_t$ is still $\mathbf{x}_t$-fundamental although $\mathbf{C}(z)$ is noninvertible. For simplicity, we do not deal with this latter possibility in the present paper and in what follows we speak about roots "outside the unit circle", meaning "outside and/or on the unit circle".

Example 1 Consider a very simple example of a univariate MA, i.e. $N = q = 1$,

$$x_t = (1 - bL)u_t, \quad u_t \sim \text{w.n.}(0, 1), \quad t \in \mathbb{Z}, \tag{5}$$

and assume that $|b| > 1$. According to condition 1, u_t is x_t-nonfundamental. Now assume to generate a process $\{\tilde{u}_t, t \in \mathbb{Z}\}$ such that

$$\tilde{u}_t = (1 - \tilde{b}L)^{-1}x_t = \sum_{k=0}^{\infty} \tilde{b}^k x_{t-k}, \quad t \in \mathbb{Z}, \tag{6}$$

where $\tilde{b} = b^{-1}$. Since $|\tilde{b}| < 1$, convergence is guaranteed and moreover $\tilde{u}_t$ is a white noise process with variance b^2. Multiplying both sides of (6) by $(1 - \tilde{b}L)$ we get

$$x_t = (1 - \tilde{b}L)\tilde{u}_t, \quad \tilde{u}_t \sim \text{w.n.}(0, b^2), \quad t \in \mathbb{Z}, \tag{7}$$

and in this case $\tilde{u}_t$ is x_t-nonfundamental. By construction x_t has always variance $(1 + b^2)$ so both (5) and (7) are valid MA representations of x_t in terms of second moments. However, the two representations do not yield the same description of x_t when we are also concerned about the impact of the underlying white noise process.

Example 2 Consider the case $N = q = 2$

$$x_{1t} = (1 - aL)u_{1t}, \quad x_{2t} = (1 - bL)u_{2t}, \quad t \in \mathbb{Z}.$$

From condition 1 we need $|a| < 1$ and $|b| < 1$ in order to have fundamentalness. Consider instead the case $N = 2$ and $q = 1$:

$$x_{1t} = (1 - aL)u_t, \quad x_{2t} = (1 - bL)u_t, \quad t \in \mathbb{Z}.$$

Now we have to consider condition 2 and we see that u_t is $\mathbf{x}_t$-fundamental for any complex number a and b provided just that $a \neq b$. In this sense condition 2 is a milder condition than 1 to be satisfied.

In the first part of the paper (Sections 4 to 7) we consider many economic models that imply squared systems. We show that in models of this kind nonfundamentalness is a generic feature that cannot be neglected. In the second part of the paper (Sections 8 and 9) we consider factor models as reduced rank models that always satisfy condition 2 and we show that in this case nonfundamentalness is a nongeneric problem.

3 A critique of Structural VARs

In order to validate and test the implications of a specific economic model, the standard econometric practice consists in assuming a finite lag approximation of (4) and, given a panel of observed data $\mathbf{x}_t$, aims at estimating the structural shocks $\mathbf{u}_t$ and the entries of $\mathbf{C}(L)$, i.e. the impulse response functions. VAR literature considers small size ($N \leq 10$) systems with $N = q$. This is due both to unavailability of data on some of the variables that appear in the economic model of reference, and to the difficulty of estimating a large size VAR. Model (4) is then estimated by means of the finite lag VAR

$$\mathbf{D}(L)\mathbf{x}_t = \mathbf{u}_t, \quad \mathbf{u}_t \sim \text{w.n.}(\mathbf{0}, \mathbf{\Gamma}_0^u), \quad t \in \mathbb{Z}, \tag{8}$$

where $\mathbf{D}(L)$ is a $N \times N$ finite lag polynomial matrix such that $\det \mathbf{D}(z)$ has roots only outside the unit circle, i.e. (8) is a causal VAR.[3]

Once (8) is estimated, identification of the structural shocks is accomplished by imposing on the estimated residuals, say $\widehat{\mathbf{v}}_t$ some additional restrictions derived from economic theory. Indeed, $\widehat{\mathbf{v}}_t$ coincide with the structural shocks only up to an istantaneous rotation (i.e. an

[3]Simple information criteria as Schwarz's or Akaike's are enough to determine the optimal lag.

orthogonal transformation). Consider a $N \times N$ transformation $\mathbf{R}$, such that $\mathbf{RR}' = \mathbf{I}$, then the spaces generated by $\widehat{\mathbf{v}}_t$ and $\mathbf{R}\widehat{\mathbf{v}}_t$ always coincide. The way we fix the $N(N-1)/2$ degrees of freedom of $\mathbf{R}$ is based on the underlying economic theory (some examples will be considered in the following when dealing with specific models). Once $\mathbf{R}$ is fixed we have the estimated structural shocks $\widehat{\mathbf{u}}_t = \mathbf{R}\widehat{\mathbf{v}}_t$. A VAR model plus the identifying specification of an orthogonal transformation $\mathbf{R}$ is called *Structural* VAR (SVAR).[4]

Whatever the identification scheme used, the estimated and identified $\widehat{\mathbf{u}}_t$ is $\mathbf{x}_t$-fundamental, given that for any t, $\mathcal{H}_t^{\widehat{u}} = \mathcal{H}_t^{\widehat{v}} = \mathcal{H}_t^{\varepsilon^x}$, where $\boldsymbol{\varepsilon}_t^x$ are the innovations of $\mathbf{x}_t$ which by definition are $\mathbf{x}_t$-fundamental. However, as soon as standard microeconomic assumptions are relaxed (e.g. by introducing learning-by-doing dynamics, as in the example in the next Section, or heterogeneous information, as in the examples in Section 5), small size economic models are more likely to be driven by structural shocks that are $\mathbf{x}_t$-nonfundamental. Indeed, when $N = q$ condition 1 can be easily violated. In this case estimating a causal VAR as (8) would never lead to recover the space generated by the structural shocks.

As shown in (3), nonfundamentalness can be restated as a case where the agents' information space is larger than the econometrician's one. Indeed, since in economic models $\mathbf{u}_t$ affects the whole economy, considering small datasets as in SVARs may lead to a wrong estimation of the structural shocks. For example, when agents have expectations about future variables, they can use additional information contained in their economic environment to form such expectations, while the econometrician can make use of just a limited amount of information. We can therefore enlarge the econometrician's information space by considering larger datastes which are nowdays more available than they used to be. While the number of shocks in the reference economic model remains small ($q \sim 5$), we can increase the number of observed variables and consider models with $N > q$. Very large datasets used in factor analysis ($N \sim 100$) are a good proxy of the whole economy. In this case a non-generic condition as condition 2 has to be satisfied to guarantee fundamentalness. By principal components methods we are able to estimate the space generated by $\mathbf{u}_t$ also in the case $N \to \infty$. Finally, we can again impose on the estimated white noise economic meaningful restrictions as in SVARs and identify the structural shocks.

Remark 2 (Forecasting) Nonfundamentalness is a problem only for the estimation of structural models. When instead we use VARs just for forecasting, we are not concerned about nonfundamentalness since in this case we are not interested in recovering the structural shocks, but we just care about an estimate of the innovations' space. Fundamental white noise arise naturally with linear prediction, being the prediction error $\mathbf{e}_t = \mathbf{x}_t - \text{Proj}\,(\mathbf{x}_t|\mathbf{x}_{t-1}, \mathbf{x}_{t-2}, \ldots)$, by construction, $\mathbf{x}_t$-fundamental. Therefore, when estimating an ARMA with forecasting purposes, the MA matrix component is always chosen to be invertible.

[4]Before 1980s identification was obtained by imposing restrictions directly on the coefficients of $\mathbf{D}(L)$. However, Sims (1980) dubbed them as "incredible" and proposed to put weaker identifying restrictions by fixing an orthogonal transformation $\mathbf{R}$ of the estimated innovations (see Watson, 1994, for a survey on SVARs).

4 Why does nonfundamentalness matter?

Nonfundamentalness appears in the econometric literature in two ways: endogenously or exogenously. In the first case the econometric model derived from economic theory is by definition nonfundamental – this is the case of permanent income models (see Blanchard and Quah, 1993; Fernández-Villaverde et al., 2007) and rational expectations (see Hansen and Sargent, 1980) – while in the second case it is the way in which the dynamics of exogenous variables is specified which makes the model fundamental or not. We start with an example of this latter case by Lippi and Reichlin (1993).

Our intention is to review the development of the issue of nonfundamentalness under a historical perspective. Therefore, we start the excursus from the work which represents the origin of the debate on nonfundamentalness, i.e. Lippi and Reichlin (1993) (LR henceforth). In a comment to a well known SVAR model by Blanchard and Quah (1989) (BQ henceforth), LR highlight the possible existence of nonfundamental white noises that, although not recoverable with a VAR, may still give rise to economic meaningful models. Both these works take, as a starting point, the following economy-wide model by Fischer (1977):

$$\begin{aligned} y_t &= m_t - p_t + a\theta_t, \qquad & y_t &= n_t + \theta_t, \\ p_t &= w_t - \theta_t, & w_t &= w^*, \qquad t \in \mathbb{Z}, \end{aligned}$$

where y_t, n_t, and θ_t denote the logs of output, employment, and productivity; w_t, p_t and m_t are the logs of nominal wage, price level, and money supply; $a\theta_t$ is investment demand with $a > 0$. In the last equation nominal wages at time t are set to a value w^* such that the expectation at time $t-1$ of employment at time t equals full employment. The evolution of money supply and productivity are given by:

$$m_t = m_{t-1} + u_t^d, \qquad \theta_t = \theta_{t-1} + d(L)u_t^s, \quad t \in \mathbb{Z}.$$

There are two types of uncorrelated structural shocks, one that has a permanent effect on output through productivity (u_t^s), while the other (u_t^d) has not. The former can be interpreted as a supply shock, while the latter as a demand shock. The model restricted to output growth rate, $(1-L)y_t$, and unemployment, U_t, has the structural form

$$\begin{bmatrix} (1-L)y_t \\ U_t \end{bmatrix} = \begin{bmatrix} (1-L) & d(L) + (1-L)a \\ -1 & -a \end{bmatrix} \begin{bmatrix} u_t^d \\ u_t^s \end{bmatrix} = \mathbf{C}(L) \begin{bmatrix} u_t^d \\ u_t^s \end{bmatrix}, \quad t \in \mathbb{Z}. \tag{9}$$

The only difference between the models by BQ and by LR is in the effect of the supply shock on output growth rate, i.e. $d(L)$. The model by BQ assumes no dynamics in productivity except for the instantaneous response to the supply shock, therefore they assume $d(L) = 1$. The model by LR assumes learning-by-doing dynamics such that the coefficients d_k of the $d(L)$ polynomial sum to 1, therefore in their model the rate of increase of productivity at time $t+k$ is $d_k u_{t-k}^s$. We now review in detail the implications of these two choices.

4.1 Fundamentalness in Blanchard and Quah (1989)

BQ estimate the following VAR

$$\mathbf{D}(L)\begin{bmatrix} \Delta y_t \\ U_t \end{bmatrix} = \begin{bmatrix} u_t^d \\ u_t^s \end{bmatrix}. \tag{10}$$

The structural shocks $\mathbf{u}_t = (u_t^d \ u_t^s)'$ are thus identified starting from the estimated white noise of a reduced form VAR by imposing long-run neutrality of the demand shock on y_t, i.e. $\mathbf{C}_{11}(1) = 0$. The inverse representation of (10) is then (9) when $d(L) = 1$. BQ obtain the following impulse responses: the effect of the demand shock is hump-shaped for both variables, while the effect of the supply shock on output increases steadily over time before reaching a plateau (solid lines in Figure 1).

Notice that the issue of nonfundamentalness is always present when dealing with VAR models, even when it is not explicitly mentioned as in the work by BQ. Indeed all their procedure is correct provided that $\mathbf{C}(L)$ is invertible. From (9), with the condition $d(L) = 1$, we have that $\det \mathbf{C}(z) = 1$, and condition 1 is trivially satisfied. Therefore the SVAR in (10) is a correct representation of the economy if $d(L) = 1$.

4.2 Nonfundamentalness in Lippi and Reichlin (1993)

LR assume a nontrivial dynamics for productivity evolution. This simple and very realistic assumption generates a variety of other possible impulse responses. Indeed, in this case $\det \mathbf{C}(z) = d(z)$, therefore fundamentalness of $\mathbf{u}_t$ is no more automatically guaranteed unless we impose additional restrictions on the process of diffusion of technology $d(L)$. However, economic theory does not provide sufficient restrictions on θ_t in order to satisfy condition 1. For instance, the typical case of learning-by-doing characterizing the diffusion of technological innovations can be modeled by assuming a bell-shaped pattern for the coefficients d_k, which generates an S-shaped long-run impulse response of the output growth rate to a supply shock. LR show that such a choice may imply that some roots of $\det \mathbf{C}(z)$ are inside the unit circle. The bottom line of the work by LR consists in the possibility of producing econometric representations of an economically sensible model in which the standard assumption of fundamentalness is violated.

Using the same data as in BQ, LR first estimate a VAR and then they invert it to get its MA representation. By moving inside the unit circle the roots of the MA polynomial which are by definition outside the unit circle, they generate different nonfundamental shocks and their corresponding impulse responses.[5] Some of the impulse responses obtained by LR are immediately rejected as implausible, while others can be given an economic interpretation. Figure 1 compares the impulse responses obtained by BQ (solid line) and the impulse responses which LR obtain from two different nonfundamental representations. While one of the experiments generates responses to nonfundamental shocks which do not substantially differ from responses to fundamental shocks, in the other nonfundamental case the shape of

[5] The procedure adopted is explained in detail in the Appendix and is based on Blaschke matrices. These are complex-valued filters which take the zeroes of a MA polynomial from outside to inside the unit circle. The main property of Blaschke transformations is that they take an orthonormal white noise into an orthonormal white noise: this ensures that the requirement of uncorrelated structural shocks is fulfilled also in the case of nonfundamental processes.

the responses is considerably different from the fundamental case. Indeed the responses to the supply shock can be interpreted as responses to a technology shock which does not have an instantaneous one-to-one impact on the variables of interest, while the response of output to the demand shock exhibits a shift in the lag structure. Also the variance decomposition changes ascribing less importance to the demand shock than in the fundamental case.

In general, the literature does not provide support for fundamentalness, so that all MA models that fulfill the same economic statements but are nonfundamental are ruled out with no justification. Although skeptical about the economic importance of nonfundamentalness, Blanchard and Quah (1993) recognize that we cannot neglect this problem an propose another example of nonfundamentalness based on the permanent income Friedman-Muth model. Income y_t is decomposed in a permanent part y_{1t} and a transitory part y_{0t} which are independently affected by uncorrelated shocks

$$(1-L)y_{1t} = u_{1t}, \qquad y_{0t} = u_{0t}, \quad t \in \mathbb{Z}.$$

If consumption c_t follows the permanent income hypothesis, as in Hall (1978), we have: $(1-L)c_t = u_{1t} + (1-\beta)u_{0t}$ where $\beta \in (0,1)$ is the agent discount factor. Therefore, we have

$$\begin{bmatrix} (1-L)y_t \\ (1-L)c_t \end{bmatrix} = \begin{bmatrix} 1 & 1-L \\ 1 & 1-\beta \end{bmatrix} \begin{bmatrix} u_{1t} \\ u_{0t} \end{bmatrix} = \mathbf{C}(L)\mathbf{u}_t, \quad t \in \mathbb{Z}.$$

In this case $\det \mathbf{C}(z) = (z-\beta)$ and hence it has the only root in $z=\beta$, which by definition lies inside the unit circle. The above econometric model representing the permanent income model is nonfundamental. Permanent and transitory components of income are not recoverable just by considering only income and consumption.

The previous example is a typical case of endogenous nonfundamentalness, in that this property does not depend on any exogenous variable, but it is just a property of the model that cannot be eliminated. On the other side the model proposed by LR is a case in which nonfundamentalness is exogenously generated by the way in which the technological shock hits the economy. However, exogeneity is not a good reason for considering nonfundamentalness an innocuous problem. Indeed, as we just showed, we can generate nonfundamental but meaningful economic models, that SVARs cannot identify. Unless we knew the whole economic environment, we should take into account all the possible econometric representations including nonfundamental ones.

Both examples in this Section show how nonfundamental shocks can arise even in very simple models where no expectations are present. This is instead the subject of the following Section.

5 Rational expectations models

Hansen and Sargent (1980) introduced the concept of nonfundamentalness while trying to set up a method for formulating and estimating dynamic linear econometric models with rational expectations. In these models, the problem lies with the fact that estimation is usually run by estimating agents' decision rules jointly with the model of the stochastic process they face, subject to the restrictions implied by the rational expectations rules. These in turn imply that agents observe and respond to more data than those the econometrician possesses, i.e. the agents' information space is larger than the econometrician's one. Hansen and Sargent (1980) express the problem as follows:

> "[...] the dynamic economic theory implies that agents' decision rules are *exact* (non-stochastic) functions of the information they possess about the relevant state variables governing the dynamic process they wish to control. The econometrician must resort to *some* device to convert the exact equations delivered by economic theory into inexact (stochastic) equations susceptible to econometric analysis".

Example 3 To fix ideas, let us take the simple example by Hansen and Sargent (1991). Suppose that one economic variables w_t, representing the true process, is generated by a fundamental moving average process, while another one x_t, representing the estimated process, is made of expectational variables. Namely,

$$\begin{aligned} w_t &= u_t - \theta u_{t-1} = \widetilde{C}(L)u_t, \\ x_t &= \mathrm{E}_t\left[\sum_{\tau=0}^{\infty} \beta^{\tau} w_{t+\tau}\right] = (1-\beta\theta)u_t - \theta u_{t-1} = C(L)u_t, \quad t \in \mathbb{Z}. \end{aligned}$$

The only root of $C(z)$ is $(1-\beta\theta)/\theta$ which can be inside the unit circle even if $\widetilde{C}(z)$ has its root outside the unit circle. If only x_t is available to the econometrician then he may not be able to recover the structural shock u_t that generates w_t.

5.1 Heterogeneous information

A recent strand of literature studies the characteristics of the equilibria in dynamic rational expectations models when the assumption of homogeneous information across agents is relaxed in favor of symmetric information. Representative of this literature are the works by Kasa (2000), Kasa et al. (2006), and Rondina (2008). In the heterogeneous information setting, nonfundamental models correspond to nonrevealing equilibria. The mechanism at work in these models is the following: agents do not directly observe the structural shocks and the equilibrium price is not fully revealing of the true state of the economy. In this case, the formation of agents' rational expectations involves a component related to the "average" market expectation, which in turn implies "forecasting the forecast of others". In other words, heterogeneous information breaks the law of iterated expectations and gives rise to higher order beliefs. The assumption of consistency between beliefs and outcomes, implied in turn by an infinite regression in expectations, produces or may produce nonfundamental MA models. In other words, nonfundamentalness is linked to nonrevealing equilibria, because in order to prevent the aggregate price to be a sufficient statistic of the state of the economy in equilibrium, the model must be such that agents cannot retrieve structural shocks from observations, therefore the model must be represented by a nonfundamental MA.

By taking into account the difference between fundamental and nonfundamental shocks, these models are able to explain puzzles which standard theory does not account for. Figure 2 reports the impulse response function of asset prices to a shock in market fundamentals (e.g. dividends) in both the full-information standard asset pricing model and in the dynamic asset pricing model with persistent heterogeneous beliefs developed by Kasa et al. (2006). The response of asset prices in the heterogeneous information case, where the MA model is nonfundamental, is more than twice as large as the standard response at impact, and the effects are persistent. This explains the empirically observed persistence in aggregate price dynamics and asset prices systematic violation of the linear present value model and standard variance bounds. Figure 3 shows how in the model by Rondina (2008), where nonfundamentalness arises endogenously via informational heterogeneity, the equilibrium aggregate price might underreact to structural aggregate technology shocks and might not allow to recover them. The incomplete information response plotted in Figure 3 corresponds to the case in which the incentive for firms to coordinate price adjustments is strong enough to turn the MA component from fundamental to nonfundamental. In this case the effect of a productivity shock on the aggregate price is dampened at impact, which explains the propagation of transitory shocks throughout the economy.

5.2 Forward looking systems

Finally, let us outline an example of how forward-looking systems with rational expectations may give origin to nonfundamental representations. What follows is taken from a recent work by Brock et al. (2008), where the authors analyze the role of rational expectations in the framework of frequency domain analysis of linear systems with feedback control rules. They show that by means of an appropriate choice of the control, e.g. monetary policy, it is possible to take the roots of the MA polynomial outside the unit circle, thereby turning a nonrevealing equilibrium into a revealing one (and vice versa).

Formally, a forward-looking system with controls is written as

$$\mathbf{D}_0\mathbf{x}_t = \beta \mathrm{E}_t(\mathbf{x}_{t+1}) + \mathbf{D}(L)\mathbf{x}_{t-1} + \mathbf{P}(L)\mathbf{c}_t + \mathbf{v}_t, \quad t \in \mathbb{Z},$$

where $\mathbf{x}_t$ are the state variables, $\mathbf{c}_t$ are the control variables and $\mathbf{v}_t = \mathbf{W}(L)\mathbf{u}_t$ with $\mathbf{u}_t$ being the structural shocks. A generic linear feedback rule is written as $\mathbf{c}_t = \mathbf{K}(L)\mathbf{x}_{t-1}$. Finally, we denote with $\mathbf{x}_t = \mathbf{C}(L)\mathbf{u}_t$ the equilibrium MA representation of the system. The key point is that $\mathbf{C}(L)$ depends on the choice of the control rule, i.e. on $\mathbf{K}(L)$. Indeed, the choice of different control rules has an impact on the spectral density matrix of the state variable $\mathbf{x}_t$, which is

$$\mathbf{\Sigma}^x(\theta) = \frac{1}{2\pi}\mathbf{C}(e^{-i\theta})\mathbf{\Sigma}^u(\theta)\mathbf{C}(e^{i\theta})', \quad \theta \in [-\pi, \pi],$$

$\mathbf{\Sigma}^u(\theta) = \mathbf{I}$ being the spectral density matrix of the structural shocks. The control enters the expression for $\mathbf{C}(e^{-i\theta})$ as follows

$$\mathbf{C}(e^{-i\theta}) = \mathbf{D}_0 - \left[\mathbf{D}(e^{-i\theta}) + \mathbf{P}(e^{-i\theta})\mathbf{K}(e^{-i\theta})e^{-i\theta}\right]^{-1}\mathbf{W}(e^{-i\theta}), \quad \theta \in [-\pi, \pi].$$

It is possible to show that the application of a given control can have an impact on the value of $\mathbf{C}(L)$ and on the location of the roots of its determinant. This is crucial in the case of forward-looking systems when the structural shocks cannot be recovered by current and past values of the state variables. These latter constitute the policymaker or the econometrician's information set, while the agents also observe the structural disturbances and know their process $\mathbf{W}(L)\mathbf{u}_t$. However, with an appropriate choice of the feedback control, the policymaker is able to turn a nonrevealing equilibrium into a revealing one, and vice versa.

Example 4 In appendix to their paper, Brock et al. (2008) provide an example in the univariate case with $D(L) = D$, $P(L) = P$, $K(L) = K$ and $W(L) = 1 + wL$. In this case, the solution of the system is

$$C(L) = \frac{\frac{1}{\lambda_1}\left(1 + \frac{w}{\lambda_1}\right)\left(1 + \frac{\lambda_1 w}{\lambda_1 + w} L\right)}{\beta(1 - \lambda_2 L)}$$

where $1/\lambda_1$ and $1/\lambda_2$ are the roots of $(\beta - z + (D + PK)z^2)$. We have fundamentalness if

$$\left|\frac{\lambda_1 w}{\lambda_1 + w}\right| < 1.$$

For given values of D, P and w, the above condition might be not satisfied in absence of a control, while it might be satisfied by choosing an appropriate value for the control K. Figure 4 shows the impulse response function of the equilibrium aggregate price in the full-information model and in a system with a nonfundamental MA component, in which a feedback control rule (monetary policy) is designed to eliminate a problematic MA component. Interestingly, it shows that the use of a control to turn a nonrevealing (nonfundamental) equilibrium into a revealing (fundamental) equilibrium will however introduce permanent distortions.

6 Dynamic Stochastic General Equilibrium models

Recent economic research in Central Banks is based on Dynamic Stochastic General Equilibrium models as tools used for analyzing national and worldwide economies. These are analytical models based on finite difference equations with stochastic disturbances. They are used for aggregating the unobserved behavior of single agents (e.g. consumers or firms) in observed macroeconomic time series. In this Section we review a method complementary to condition 1 recently proposed by Fernández-Villaverde et al. (2007) to detect nonfundamentalness in DSGE models. It is worth discussing why we should check for nonfundamentalness before estimating a SVAR and what are the consequences in terms of validation of DSGE models if nonfundamentalness is not recognized. Indeed, the bottom line is that the whole debate on the effectiveness of SVAR techniques for DSGE model evaluation is rooted into nonfundamentalness.

6.1 Validating DSGE models by SVARs

A general DSGE model is formulated as follows:

$$\begin{aligned} \max_{\mathbf{W}_t} \quad & \mathrm{E}_t \left[\sum_{\tau=0}^{\infty} \beta^{t+\tau}\, U(\mathbf{W}_{t+\tau}) \right], \\ \text{s.t.} \quad & g(\mathbf{W}_t, \mathbf{W}_{t-1}, \ldots, \mathbf{Z}_t, \mathbf{Z}_{t-1}) \leq 0, \quad t \in \mathbb{Z}, \end{aligned}$$

where $U(\cdot)$ is a utility function and $g(\cdot)$ is a function representing economic constraints. The model includes p endogenous variables $\mathbf{W}_t$ and q exogenous variables $\mathbf{Z}_t$, which are usually modeled as functions of q serially uncorrelated orthonormal white noise processes, i.e. the structural shocks $\mathbf{u}_t$.[6] Therefore, the system contains $N = p + q$ variables $\mathbf{X}_t = (\mathbf{W}_t' \ \mathbf{Z}_t')'$.

[6] For simplicity we omit the usual distinction between non-predetermined and predetermined enodgenous variables as conclusions do not change.

Let us indicate with small letters the difference between the log of the variables and their non-stochastic steady state. We have the linearization of the model

$$\mathbf{w}_t = \mathbf{T}(L)\mathbf{z}_t\,, \qquad \mathbf{S}(L)\mathbf{z}_t = \mathbf{u}_t, \quad t \in \mathbb{Z}. \tag{11}$$

The system can be transformed into a state space form by defining the state variables as $\mathbf{f}_t = (\mathbf{z}'_t, \dots, \mathbf{z}'_{t-s})$ where s is the maximum degree between $\mathbf{S}(L)$ and $\mathbf{T}(L)$ (see Giannone et al., 2006, for details). The dimension of $\mathbf{f}_t$ is $r = q(s+1)$, with $q \leq r \leq N$. Therefore, we have

$$\begin{aligned} \mathbf{x}_t &= \mathbf{\Lambda}\mathbf{f}_t\,, \\ \mathbf{A}(L)\mathbf{f}_t &= \mathbf{B}\mathbf{u}_t, \quad t \in \mathbb{Z}, \end{aligned} \tag{12}$$

where $\mathbf{\Lambda}$ is $N \times r$ and $\mathbf{B}$ is $r \times q$. We have a system with an N-dimensional vector of observable variables $\mathbf{x}_t$ and a q-dimensional vector of structural shocks $\mathbf{u}_t$, such that $\mathbf{u}_t$ is an orthonormal white noise process.

The procedure, used for example in a couple of recent papers by Chari et al. (2008) and by Christiano et al. (2006), for assessing the reliability of SVARs as a tool to discriminate among competing models, is the following:

1. consider a DSGE model (e.g. a real business cycle model or a New-Keynesian model);
2. reformulate it in a state space form as (12) usually obtained by log-linearizing around the non-stochastic steady state;
3. calibrate the parameters of the state space form by choosing values from existing literature or by minimizing the distance with theoretical impulse responses (see e.g. Christiano et al., 2005);
4. by repeatedly extracting shocks from a suitable random distribution generate new data from the state space model with the parameters of the previous step;
5. using the simulated data of previous step, estimate a SVAR, and compute the impulse responses;
6. having repeated the last two steps thousands of times, we have a simulated confidence interval for the impulse responses;
7. check if the theoretical impulse responses of step 3 lie inside the simulated confidence intervals and, if this is the case, we can say that SVARs are useful tools for discriminating among competing economic models.

Once we have assessed the usefulness of SVARs, we can validate DSGE models by means of the following additional steps:

8. estimate the impulse responses on observed data together with their confidence intervals obtained with boostrap methods;
9. if the theoretical impulse responses lie inside the estimated confidence intervals, we accept the underlying economic model, while if they do not we reject it.

6.2 Nonfundamentalness in DSGE models

The described procedure suffers of two main potential problems. First, we have to assume that the DSGE model has a finite lag VAR representation. This may be not true in general and it may introduce a bias in the estimated impulse responses. Second, even if conditions for a finite VAR representation hold (see Ravenna, 2007), observations for many state variables (usually stocks as e.g. capital) are typically not available when validating DSGE models when using real data (steps 8 and 9). Therefore, it is not possible to estimate the same impulse responses as the simulated ones since some of the variables of the underlying DSGE have to be omitted. Either when a finite VAR model is not valid or when some varibles are omitted, we have to consider VARMA models and therefore we face the possibility of having a noninvertible MA component, i.e. of having nonfundamental shocks.

A fiscal policy example (see Pagan, 2007), although related to noninvertibility rather than to nonfundamentalness, illustrates the argument. Let x_t be the primary deficit, and the level of debt is defined as a gap relative to its desired equilibrium value. Debt accumulates as $(1-L)d_t = x_t$ where we set the interest rate on past debt to zero. In order to stabilize debt, we need a fiscal rule that relates to the past debt level and responds to an output gap y_t, i.e.

$$(1-L)d_t = x_t = ad_{t-1} + cy_t + u_t \quad \text{with} \quad a < 0, \quad t \in \mathbb{Z}.$$

Typically, we drop debt from this AR model as it is not observed, thus, we need to solve the previous equation for d_t and substitute it in the fiscal policy equation, obtaining

$$x_t = (1+a)x_{t-1} + c(1-L)y_t + (1-L)u_t, \quad t \in \mathbb{Z}.$$

This is no more an AR but an ARMA where the MA part $(1-L)u_t$ has its root in $z=1$, thus it is not invertible. Indeed, Favero and Giavazzi (2007) show that omitting the level of debt from the VAR can result in biased estimates of the effects of fiscal policy shocks. In particular, if debt dynamics are unstable the impulse response functions will eventually diverge.

Given the presence of expectations in DSGE models, it is not unlikely to face a problem of omitted variables and therefore of nonfundamentalness. It would be advisable to have a criterion that allows to identify nonfundamentalness by starting from the structural parameters of the DSGE model under consideration, Such parameters are contained in the matrices $\mathbf{A}(L)$, $\mathbf{B}$, and $\mathbf{\Lambda}$ and define preferences, technology, and, in general, economic shocks. If nonfundamentalness is immediately recognized in a DSGE, the entire procedure of validation through SVARs is invalid and alternative validation methods should be considered.

What we are looking for is the condition under which we can write a VAR as a linearized solution of a DSGE model as (12) (see Hannan and Deistler, 1988; Fernández-Villaverde et al., 2007). In the typical small size models $\mathbf{\Lambda}$ has maximum rank N so that $r=N$ and we can write $\mathbf{f}_t = \mathbf{\Lambda}^{-1}\mathbf{x}_t$. Plugging $\mathbf{f}_t$ into the second equation of (12) we get

$$(\mathbf{I} - \mathbf{\Lambda}\mathbf{A}\mathbf{\Lambda}^{-1}L)\mathbf{x}_t = \mathbf{\Lambda}\mathbf{B}\mathbf{u}_t, \quad t \in \mathbb{Z}.$$

This is a causal VAR representation of the DSGE only if the roots of $\mathbf{\Lambda}\mathbf{A}\mathbf{\Lambda}^{-1}$ lie all outside the unit circle and, in this case, we can use VARs to validate the DSGE model. This condition, is nothing else but the counterpart of condition 1 for state space models in the full rank case

$N = r$. However, in many cases DSGE models consider a large number of variables and therefore are likely to have reduced rank $r < N$ so that $\mathbf{\Lambda}^{-1}$ is not defined. In Section 8 we consider a different condition for nonfundamentalness and an alternative validation method for reduced rank systems, while in the next Section we make a preliminary point on the value of additional information for ruling out nonfundamental models.

7 The value of additional information

Giannone and Reichlin (2006) show the value added of enlarging the econometricians' information set when dealing with nonfundmental models. They propose a criterion to detect nonfundamentalness based on Granger causality and, as an example, they consider a well known SVAR firstly estimated by Galí (1999). This model can be derived from very different DSGE models with opposite impications such as real business cycle models or New-Keynesian models, namely we have

$$\left[\begin{array}{c}(1-L)a_t\\(1-L)\ell_t\end{array}\right] = \mathbf{C}(L)\left[\begin{array}{c}u_t^s\\u_t^d\end{array}\right], \quad t\in\mathbb{Z}, \tag{13}$$

where a_t is the log of aggregate labor productivity and ℓ_t is the log of aggregate labor supply (e.g. hours worked per capita). There are two structural shocks: a technological shock u_t^s and a shock u_t^d which is neutral for productivity in the long-run, being thus interpretable as a labor income (or demand) shock or a monetary shock. Let us define $\mathbf{x}_t = ((1-L)a_t \;\; (1-L)\ell_t)'$ the vector of observable variables which we augment with other variables $\mathbf{x}_t^*$, so that (13) for the larger system becomes

$$\left[\begin{array}{c}\mathbf{x}_t\\\mathbf{x}_t^*\end{array}\right] = \left[\begin{array}{cc}\mathbf{C}(L) & 0\\\mathbf{C}^*(L) & \mathbf{\Psi}(L)\end{array}\right]\left[\begin{array}{c}\mathbf{u}_t\\\mathbf{u}_t^*\end{array}\right], \quad t\in\mathbb{Z},$$

with $\mathbf{u}_t^*$ as additional structural shocks orthogonal to $\mathbf{u}_t$. If $\mathbf{u}_t$ is $\mathbf{x}_t$-fundamental, then there exists a one-sided filter $\mathbf{D}(L)$ such that $\mathbf{u}_t = \mathbf{D}(L)\mathbf{x}_t$, therefore

$$x_{it}^* = \mathbf{C}_{i\cdot}^*(L)\mathbf{D}(L)\mathbf{x}_t + \mathbf{\Psi}_{i\cdot}(L)\mathbf{u}_t^* \qquad \text{for } i=1,\dots,N, \quad t\in\mathbb{Z},$$

where $\mathbf{C}_{i\cdot}^*$ indicates the i-th row of $\mathbf{C}^*$. Hence, each x_{it}^* depends only on the past of $\mathbf{x}_t$ and does not incorporate any further information useful for forecasting $\mathbf{x}_t$, i.e. none of the x_{it}^* Granger causes $\mathbf{x}_t$. This result was firstly introduced by Forni and Reichlin (1996). It follows that nonfundamentalness can be detected empirically by checking whether the variables of interest $\mathbf{x}_t$ are weakly exogenous with respect to potentially relevant additional blocks of variables, where the variables contained in the additional blocks are likely to be driven by shocks which are common to the variables belonging to the block of interest. In the model above, Giannone and Reichlin (2006) consider as additional variables labor productivity and labor input at sectoral level and they indeed reject the hypothesis of weak exogeneity, thus giving a clue for nonfundamentalness. Their result shows that nonfundamental white noise processes are different from the fundamental ones estimated in the SVAR, or in other words the shocks estimated in the SVAR are not the structural shocks. Therefore, nothing can be said about the dispute between real business cycle and New-Keynesian models by looking only at labor productivity and labor input as it is usually done in SVAR literature.

8 Large cross-sections for structural identification

Nonfundamentalness is ultimately a problem of missing information. As we have seen, if we have nonfundamentalness the inverse of a MA involves future observations. Indeed, this is in principle a first approach we might take: by estimating a VAR we will never be able to retrieve $\mathbf{u}_t$, since for some $k > 0$ we would need $\mathbf{x}_{t+k}$, but we could still estimate $\mathbf{u}_{t-k}$. What if we want to retrieve the contemporaneous structural shocks? We need to enlarge the econometrician's information set in some other manner which is not including future observations. The alternative to the time dimension is the cross-section dimension.

Recently some attempts of supplying the missing information in VARs have been done. Blanchard and Perotti (2002) augment their model by means of Instrumental Variables, while Mountford and Uhlig (2009) consider a ten dimensional VAR but identify only three shocks, i.e. impose a reduced rank impulse response matrix. However, there are econometric models which are more suitable than VARs when we need to deal with very large datasets. The main candidates are: Bayesian VARs (see e.g. De Mol et al., 2008; Bańbura et al., 2010), Global VARs (see e.g. Pesaran et al., 2004; Di Mauro et al., 2007), Factor Augmented VARs (see e.g. Bernanke et al., 2005), and Dynamic Factor models (Forni et al., 2000, 2009). In this Section we consider factor models and outline how they are built and how they deal with nonfundamentalness. In this case, the small number of structural shocks typically implied by economic models is represented by few dynamic common factors which have a pervasive impact on all the variables of a large dataset. Such large panels are a good proxy of the whole economy. Thus, by using few shocks and many variables we are considering a consistent representation of typical economic models.

8.1 Reduced rank, Dynamic Factor models, and DSGE models

Giannone et al. (2006) and Boivin and Giannoni (2006) provide the motivation for considering a factor structure in validating DSGE models. Typical theoretical macroeconomic models have few shocks driving the business cycle, e.g. only one technology shock in first generation real business cycle models, two or three in second generation ones. With our notation we have $N > q$ and we say that these models have reduced stochastic rank. Usually in DSGE models also measurements errors are considered, and in this case it can be shown that the model can have a factor structure, since factor models by their own nature separate out measurement errors. Indeed, the spectral density matrix of the observed variables can be decomposed into two orthogonal parts: the spectral density of the common component, of reduced rank, that contains all the relevant information of covariances (at all leads and lags), and the spectral density of the idiosyncratic component, of full rank, that represents non-correlated or mildly correlated measurement errors. Therefore, factor models seem to be a good alternative tool to validate DSGE models, as formally discussed in this Section.

Let us start from model (4)

$$\mathbf{x}_t = \mathbf{C}(L)\mathbf{u}_t, \quad t \in \mathbb{Z}. \tag{14}$$

The dynamic rank of $\mathbf{x}_t$ (i.e. the rank of its spectral density matrix) is q, i.e. is the number of structural shocks. In general, economic theory imposes a small value for q, and indeed this is the case for large macroeconomic datasets, where empirically we find $q < N$, i.e. the N variables are collinear. Independently from the fact that $\mathbf{u}_t$ is $\mathbf{x}_t$-fundamental or not, in order to estimate a SVAR we would need to invert a reduced rank covariance matrix, which is obiously not feasible. There are two alternatives to deal with reduced ranks: either we estimate a SVAR only on full-rank blocks of variables, or we add measurement errors. While in the former case nonfundamentalness in the subpanels may still be a problem to solve, in the latter case we both eliminate the collinearity among variables and we make nonfundamentalness a nongeneric problem by estimating a dynamic factor model on the data. In the following, we will focus on this latter case (see Giannone et al., 2006, for the other alternatives).

When adding orthogonal measurement errors $\boldsymbol{\xi}_t$, we lose collinearity of the variables and we can write (14) for a covariance stationary process $\mathbf{x}_t$, as a dynamic factor model

$$\mathbf{x}_t = \mathbf{C}(L)\mathbf{u}_t + \boldsymbol{\xi}_t = \boldsymbol{\chi}_t + \boldsymbol{\xi}_t, \quad t \in \mathbb{Z}, \tag{15}$$

where $\mathbf{u}_t$ is the q-dimensional orthonormal vector of white noise common (structural) shocks; $\boldsymbol{\chi}_t$ is common to all variables; $\boldsymbol{\xi}_t$ is an idiosyncratic N-dimensional process, such that ξ_{it-k} is orthogonal to u_{jt} for any integer i, j, and k. We need three assumptions for identifying the factor structure (see Forni et al., 2000):

1. $\mathbf{x}_t$ has rational spectral density;
2. for all frequencies $\theta \in [-\pi, \pi]$, the q largest eigenvalues of the spectral density of $\mathbf{x}_t$ diverge as $N \to \infty$;
3. the $(q+1)$-th largest eigenvalue of the spectral density of $\mathbf{x}_t$ is bounded almost everywhere in $[-\pi, \pi]$.

These assumptions are reasonable and general enough. Indeed, measurement errors are supposed to vanish when considering linear combinations of many collinear variables, but still are allowed to have a non-diagonal covariance matrix, i.e. they can be mildly cross-sectional correlated. As a consequence, the common component $\boldsymbol{\chi}_t$ has reduced dynamic rank $q < N$, while $\boldsymbol{\xi}_t$ has full dynamic rank: this is how we treat collinearity.

If we assume that $\boldsymbol{\chi}_t$ generates a finite dimensional space, i.e. in (14) we allow only for finite lag MA loadings, then Forni et al. (2009) show that (15) has also a state space representation

$$\begin{aligned} \mathbf{x}_t &= \boldsymbol{\Lambda}\mathbf{f}_t + \boldsymbol{\xi}_t\,, \\ \mathbf{A}(L)\mathbf{f}_t &= \mathbf{B}\mathbf{u}_t, \quad t \in \mathbb{Z}, \end{aligned} \tag{16}$$

where $\mathbf{f}_t$ is r-dimensional and $r > q$. In this case the identifying assumptions 2 and 3 require that the r largest eigenvalues of the covariance matrix of $\mathbf{x}_t$ diverge as $N \to \infty$, while the $(r+1)$-th stays bounded. Once again, given these assumptions, we have a common component with reduced static rank r (i.e. the rank of the covariance matrix of $\boldsymbol{\chi}_t$) and an idiosyncratic component with asymptotically vanishing covariance that has full static rank. Hallin and Liška (2007) and Bai and Ng (2002) provide consistent criteria for estimating q and r respectively. Results obtained when applying these criteria to real data fully support the hypothesis of reduced dynamic and static ranks, thus justifying the factor approach. We call $\mathbf{f}_t$ the static factors while $\mathbf{u}_t$ are the dynamic factors which correspond to the structural shocks of the economy.

To sum up, there are at least two main advantages in imposing a factor structure as (16) on the linearized solution of a DSGE model.

1. Although we need both large cross-sections (N) in order to identify the common and idiosyncratic components, and large sample sizes (T) in order to have consistent estimates, these are perfectly realistic requirements coherent with the standard practice followed by central banks in decision making based on DSGE models.

2. The variables in (16) have a natural interpretation in DSGE models: $\mathbf{x}_t$ contains the observed variables and some proxies of the state variables which are often unobserved and can be estimated as the latent static factors $\mathbf{f}_t$. Indeed, the typical macroeconomic variables included in the panel are indicators of economic activity built by aggregation, which can be seen as linear combinations of unobserved state variables (and their lags) plus some measurement errors $\boldsymbol{\xi}_t$. It is possible then to impose structural relations between the observed $\mathbf{x}_t$ and the unobserved $\mathbf{f}_t$, i.e. to impose restrictions on $\boldsymbol{\Lambda}$.

There are two ways to estimate (16): (i) carry out a two-step estimation as in Forni et al. (2009) and impose identifying restrictions at the end; (ii) apply a one-step Quasi-Maximum Likelihood estimator with identifying restrictions as in Doz et al. (2006). Here we consider only the first method.

8.2 Fundamentalness in dynamic factor models

Why in the previous section, when considering factor models as a tool for validating DSGE models, have we not raised the issue of fundamentalness, that is pervasive when dealing with VARs? Because we can show that actually nonfundamentalness is not a generic problem in factor models, and, under reasonable assumptions, we can always guarantee that the dynamic factors $\mathbf{u}_t$ are $\mathbf{x}_t$-fundamental (see Forni et al., 2009). In factor models we always have $N > q$, therefore we need condition 2 of nonfundamentalness that generalizes condition 1 to the case of singular systems. Indeed, it is the singularity of dynamic factor models that makes the property of nonfundamentalness non generic.

Consider the case $q = 1$. If $N > 1$ we have N submatrices $\mathbf{C}_j(z)$ and, from condition 2, the representation is nonfundamental if their determinants have a common root smaller than one in modulus. Thus, if $N = q$, nonfundamentalness is generic since, if it holds in a point then, for continuity of the roots of $\mathbf{C}(z)$, it holds also in its neighborhood (see condition 1); while if $N > q$ nonfundamentalness is non generic because to have a common root we must satisfy $\binom{N}{q} - 1$ equality constraints. Intuitively, when N is large, it is highly improbable to have a common root for all submatrices $\mathbf{C}_j(L)$, as the number of restrictions we should satisfy increases with N.

To have fundamentalness we ask for $\mathbf{C}(L)$ to have maximum rank q inside the unit circle, i.e. its columns must be independent. This is equivalent to ask for heterogeneity of the impulse responses (which are the elements of $\mathbf{C}(L)$) of the N variables to the q structural shocks. In very large datasets this is very likely to happen and it is therefore reasonable to assume fundamentalness. Roughly speaking, although in principle the econometrician has a smaller information set than the agents' one (i.e. there is room for potential nonfundamentalness), he can include additional series in the system, and, if dynamic heterogeneity is guaranteed, then these series contain truly new and useful information.

To clarify this point, let us recall the permanent income model presented in Section 4 (see also Fernández-Villaverde et al., 2007). We have seen that the model is endogenously nonfundamental as the agent discount factor β, which coincides with the only root of det $\mathbf{C}(z)$ is, by definition, inside the unit circle. However, if the econometrician observes also some additional variables $\mathbf{x}_t^*$ such that $(1-L)x_{it}^* = \mathbf{b}_i(L)\mathbf{u}_t$, then $\mathbf{u}_t$ is fundamental for the whole system unless $\mathbf{C}(z)$ and $\mathbf{b}_i(z)$ have the same root, i.e. unless $\mathbf{b}_i(\beta) = \mathbf{0}$ for every variable x_{it}^* added, which is extremely unlikely. Notice that adding savings $s_t = y_t - c_t$ as an additional variable, for example, would not work, indeed

$$\begin{bmatrix} (1-L)y_t \\ (1-L)c_t \\ (1-L)s_t \end{bmatrix} = \begin{bmatrix} 1 & 1-L \\ 1 & 1-\beta \\ 0 & (\beta - L) \end{bmatrix} \begin{bmatrix} u_{1t} \\ u_{0t} \end{bmatrix} = \begin{bmatrix} \mathbf{C}(L) \\ \mathbf{b}(L) \end{bmatrix} \mathbf{u}_t, \quad t \in \mathbb{Z}.$$

The determinants of the three $q \times q$ submatrices will all have their only root in β, so it will still be impossible to recover $\mathbf{u}_t$ using only past and present observations of income, consumption and savings. This makes sense, as, being savings just a linear combination of income and consumption, they do not contain any useful information, i.e. we do not have any dynamic heterogeneity in the system.

Why can we safely make the assumptions of fundamentalness in dynamic factor models? Or, equivalently: is dynamic heterogeneity a generic feature of dynamic factor models?

Anderson and Deistler (2008) prove that if we consider the dynamic factor model (15) of a process with rational spectral density, together with the assumptions of pervasiveness of dynamic common factors and boundedness of idiosyncratic shocks as $N \to \infty$, we guarantee fundamentalness. They prove the following Theorem.

Theorem 1 (Anderson and Deistler 2008) *Consider an N-dimensional process $\boldsymbol{\chi}_t$, with rational spectral density $\boldsymbol{\Sigma}^\chi(\theta)$ whose q-largest eigenvalues diverge as $N \to \infty$, and $N > q$. Then, $\boldsymbol{\Sigma}^\chi(\theta)$ can be factorized as*

$$\boldsymbol{\Sigma}^\chi(\theta) = \frac{1}{2\pi}\mathbf{C}(e^{-i\theta})\mathbf{C}(e^{i\theta})', \quad \theta \in [-\pi, \pi],$$

where $C(z)$ is $N \times q$, rational, has no poles and no zeroes for $|z| \leq 1$ and is associated to the representation

$$\boldsymbol{\chi}_t = \sum_{k=0}^{\infty} \mathbf{C}_k \mathbf{u}_{t-k} = \mathbf{C}(L)\mathbf{u}_t, \quad t \in \mathbb{Z},$$

where $\mathbf{u}_t$ is a q-dimensional weak white noise.

This Theorem guarantees fundamentalness of the shocks $\mathbf{u}_t$ in (15). In practice, we prove below that, since N is very large, $\mathbf{u}_t$ is fundamental for the common component $\boldsymbol{\chi}_t$, while the idiosyncratic component can be neglected and considered as a vanishing measurement error.

For simplicty of estimation we consider the state-space model (16), which has also a more direct interpretation in the DSGE framework. When considering (16) we say that $\mathbf{C}(L)$ has a minimal r-dimensional state-space representation, which we indicate with the triple

$(\mathbf{\Lambda}, \mathbf{A}(L), \mathbf{B})$. The following Theorem by Anderson and Deistler (2008) proves fundamentalness of $\mathbf{u}_t$ for $\mathbf{A}(L)$ of order 1.

Theorem 2 (Anderson and Deistler 2008) *Consider a $N \times q$ rational function $\mathbf{C}(z)$ with the minimal state space representation $(\mathbf{\Lambda}, \mathbf{A}, \mathbf{B})$ with dimension $r > q$. If $N > r$ holds, then, for generic values of $(\mathbf{\Lambda}, \mathbf{A}, \mathbf{B})$, the matrix $\mathbf{C}(z)$ has full rank q for all $z \in \mathbb{C}$.*

In general, asking for the existence of a state space representation (16) is equivalent to ask for the existence of a squared-summable one-sided $r \times q$ filter $\mathbf{N}(L)$ such that

$$\mathbf{C}(L) = \mathbf{\Lambda}\mathbf{N}(L) \qquad \text{and} \qquad \mathbf{f}_t = \mathbf{N}(L)\mathbf{u}_t, \quad t \in \mathbb{Z}.$$

Asking for dynamic heterogeneity and reduced rank is equivalent to ask for $\text{rank}\,(\mathbf{\Lambda}'\mathbf{\Lambda}/N) = r$ as $N \to \infty$. Moreover, fundamentalness is equivalent to requiring left-invertibility of $\mathbf{N}(L)$, i.e. to the existence of a $q \times r$ one-sided filter $\mathbf{G}(L)$ such that $\mathbf{G}(L)\mathbf{N}(L) = \mathbf{I}_q$. Indeed, with these requirements, we can define the one-sided filter $\mathbf{S}(L) = \mathbf{G}(L)(\mathbf{\Lambda}'\mathbf{\Lambda})^{-1}\mathbf{\Lambda}'$, we have

$$\mathbf{S}(L)\mathbf{x}_t = \mathbf{G}(L)(\mathbf{\Lambda}'\mathbf{\Lambda})^{-1}\mathbf{\Lambda}'\mathbf{\Lambda}\mathbf{f}_t + \mathbf{S}(L)\boldsymbol{\xi}_t \stackrel{m.s.}{\longrightarrow} \mathbf{G}(L)\mathbf{N}(L)\mathbf{u}_t = \mathbf{u}_t \qquad \text{for } N \to \infty, \quad t \in \mathbb{Z}.$$

where convergence is in mean-square. Notice that the autoregressive polynomial $\mathbf{S}(L)$ is well defined if and only if $\mathbf{\Lambda}'\mathbf{\Lambda}$ has full rank. Moreover, $\mathbf{S}(L)$ is one-sided if and only if $\mathbf{G}(L)$ exists and is one-sided. As $N \to \infty$, $\mathbf{u}_t$ lies in the space spanned by the present and past values of $\mathbf{x}_t$, i.e. the structural shocks are $\mathbf{x}_t$-fundamental. In terms of Hilbert spaces, by recognizing that $\mathcal{H}_t^x = \mathcal{H}_t^\chi \oplus \mathcal{H}_t^\xi$, then, as $N \to \infty$, we have $\mathcal{H}_t^x \to \mathcal{H}_t^\chi = \mathcal{H}_t^u$, for any $t \in \mathbb{Z}$.

Example 5 Consider the following example from Forni et al. (2009) where we have one dynamic factor loaded with one lag. The common part of the i-th series is $\chi_{it} = (1 - c_i L)u_t$ or, equivalently,

$$\chi_{it} = \boldsymbol{\lambda}_i \mathbf{f}_t, \qquad \mathbf{f}_t = \mathbf{N}(L)u_t, \quad i = 1, \dots N, \quad t \in \mathbb{Z},$$

where $\boldsymbol{\lambda}_i$ is the i-th row of $\mathbf{\Lambda}$. It easy to see that

$$\boldsymbol{\lambda}_i = (1 \quad -c_i), \qquad \mathbf{N}(L) = \begin{bmatrix} 1 \\ L \end{bmatrix}, \quad i = 1, \dots N.$$

If $c_i \neq c_j$, i.e. we have dynamic heterogeneity, then

$$\text{rank}\,(\mathbf{\Lambda}'\mathbf{\Lambda}) = \text{rank} \begin{bmatrix} N & -\sum_i c_i \\ -\sum_i c_i & \sum_i (c_i)^2 \end{bmatrix} = 2, \quad i = 1, \dots N.$$

and we can recover the shocks from any couple of series as

$$u_t = \frac{c_j \chi_{it} - c_i \chi_{jt}}{c_j - c_i}, \quad i, j = 1, \dots N, \quad t \in \mathbb{Z}.$$

Therefore, u_t is (χ_{it}, χ_{jt})-fundamental even if $c_i > 1$ for any i, i.e. even if u_t is χ_{it}-nonfundamental. In this case $\mathbf{S}(L)$ is well defined and its two generic rows are

$$\begin{pmatrix} 1 - \frac{c_i c_j L}{c_j - c_i} & \frac{c_i^2 L}{c_j - c_i} \\ -\frac{c_j^2 L}{c_j - c_i} & 1 + \frac{c_i c_j L}{c_j - c_i} \end{pmatrix}, \quad i, j = 1, \dots N.$$

If we instead had homogeneous responses we would have $c_i = c$ for any i. In this case, rank $(\mathbf{\Lambda}'\mathbf{\Lambda}) = 1$ and clearly $\mathbf{S}(L)$ is not well defined and thus $\mathbf{u}_t$ is $\mathbf{x}_t$-nonfundamental.

Summing up, fundamentalness is equivalent to the assumptions of reduced rank (i.e. co-movements) and dynamic heterogeneity, which are perfectly reasonable when dealing with large datasets. As already pointed out, in empirical applications with large cross sections we often find evidence of reduced static and dynamic ranks, i.e. $r < N$ and $q < N$, which suggest the presence of few pervasive factors. Dynamic heterogeneity is then a reasonable property in a factor model with large cross sectional dimension N as economic variables react differently to the few structural shocks. Even if the structural shocks are fundamental for the whole system, they may not be fundamental for some subsamples of series. However, this is not a major problem in this context. Indeed, thanks to dynamic heterogeneity, the missing information due to local nonfundamentalness is completed with additional cross sectional information contained in other series. In this sense, we can consider nonfundamentalness a non generic problem for dynamic factor models.

8.3 Estimation and identification of impulse responses

Giannone et al. (2004) and Forni et al. (2009) propose a procedure to consistently estimate and identify the structural shocks and the impulse responses they generate. From (16), the impulse responses of $\boldsymbol{\chi}_t$ to a q-dimensional vector of shocks $\mathbf{u}_t$ are $\mathbf{C}(L) = \mathbf{\Lambda}\mathbf{A}(L)^{-1}\mathbf{B}$. These functions are well defined as we always assume in this paper to have causal AR polynomials, hence $\mathbf{A}(L)$ is invertible. As we can assume that the idiosyncratic component contains negligible information (e.g. it contains measurement errors), after estimating a dynamic factor model on a large panel $\mathbf{x}_t$, we can compare factor model generated impulse responses of a q-dimensional subset of series $\{\widetilde{\mathbf{x}}_t\} \subset \{\mathbf{x}_t\}$, with those obtained by estimating a q-dimensional SVAR using only $\widetilde{\mathbf{x}}_t$. In order to do this, we must recall that, when estimating a dynamic factor model, only a linear basis of the spaces spanned by the dynamic factors is identified. However, given that $\mathbf{u}_t$ is fundamental for $\boldsymbol{\chi}_t$, i.e. the structural shocks are fundamental for $\boldsymbol{\chi}_t$, we can identify them by using a contemporaneous rotation of the estimated basis of the space containing the dynamic factors. We estimate, (16) in three steps:

1. use asymptotic principal components analysis to estimate the static factors and their loadings: $\widehat{\mathbf{f}}_t$, $\widehat{\mathbf{\Lambda}}$ (see e.g. Bai and Ng, 2002);

2. fit a VAR on the estimated static factors and get the coefficients $\widehat{\mathbf{A}}(L)$ and the residuals $\widehat{\mathbf{e}}_t$;

3. apply principal components to the covariance matrix of $\widehat{\mathbf{e}}_t$ and estimate the eigenvectors matrix $\widehat{\mathbf{M}}$ corresponding to the q largest eigevalues, up to a normalization constant the estimated dynamic factors are $\widehat{\mathbf{v}}_t = \widehat{\mathbf{M}}'\widehat{\mathbf{e}}_t$ and $\widehat{\mathbf{B}} = \widehat{\mathbf{M}}$.

With this procedure we get the estimated common component $\widehat{\boldsymbol{\chi}}_t = \widehat{\mathbf{\Lambda}}\widehat{\mathbf{A}}(L)^{-1}\widehat{\mathbf{B}}\widehat{\mathbf{v}}_t$. For identifying the structural shocks we just have to determine an orthogonal matrix $\mathbf{R}$ such that the transformed impulse responses of the chosen subset $\{\widetilde{\mathbf{x}}_t\}$ of series satisfy a set of restrictions (analogous to those imposed for the same series in SVARs) implied by economic theory. The identified impulse responses and structural shocks are then $\widehat{\mathbf{C}}(L) = \widehat{\mathbf{\Lambda}}\widehat{\mathbf{A}}(L)^{-1}\widehat{\mathbf{B}}\mathbf{R}$ and $\widehat{\mathbf{u}}_t = \mathbf{R}'\widehat{\mathbf{v}}_t$ respectively.

This procedure has two advantages. First, while fundamentalness in SVARs is arbitrary, in dynamic factor models is guaranteed by two reasonable assumptions as reduced rank and dynamic heterogeneity. Second, identification is always reduced to the choice of an orthogonal matrix $\mathbf{R}$ with only $q(q-1)/2$ parameters, independently of the number of series considered, which is typically much larger than q. In contrast with the SVAR case, we do not have to impose any limitation on the size of the panel in order to estimate and identify the structural shocks.

Forni et al. (2009) carry out an empirical application of the previous procedure choosing as benchmark SVAR the model by King et al. (1991), which comprises output, consumption and investment. They include these three variables into a much larger system of other 89 variables in addition to the three of interest. They estimate a dynamic factor model with three dynamic common factors in analogy to the three shocks in the SVAR. The model is estimated for different numbers r of static factors and identified by imposing the same long-run restrictions as in the SVAR. Table 1 reports the variance decomposition results for the dynamic factor model with 15 static common factors and for the SVAR by King et al. (1991). The impulse response functions from the large system imply a larger effect of the permanent shock on output and investment than in the original SVAR. This means that the typical SVAR puzzle, concerning the small amount of investment variance explained by supply shocks in the medium-long run, might be due to the fact that the structural shocks associated with output, consumption and investment are indeed nonfundamental for the three-dimensional system and thus are not recovered by means of SVARs.

Finally, it is worth to mention at least three other recent applications of factor models to the identification of structural economic shocks. Forni and Gambetti (2010) analyze the effects of monetary policy shocks identified using a standard recursive scheme, in which the impact effects on both industrial production and prices are zero. They solve many SVAR puzzles: (i) the maximal effect on bilateral real exchange rates is observed on impact, so that the "delayed overshooting" or "forward discount" puzzle disappears; (ii) after a contractionary shock prices fall at all horizons, so that the price puzzle is not there; (iii) monetary policy has a sizable effect on both real and nominal variables. Luciani (2009) considers an economic model with five underlying structural shocks (oil price, productivity, aggregate demand, monetary policy, and housing demand) and recovers impulse responses using a factor model approach. He shows that, after the early eighties' liberalizations in housing finance, the housing demand shock has become a substantial source of business cycle fluctuations. He also analyzes the causes of the 2008 recession: results hints towards a non negligible role of monetary policy played in leading the way for the downturn in residential investment and the ensuing recession. Finally, as mentioned in Section 7, Giannone and Reichlin (2006) use factor models for analyzing the relation between labor productivity and hours worked. We review their results in the next Section where we also provide new evidence of nonfundamentalness in this bidimensional model.

9 On the relation between technology and employment

In this Section we apply the method of the previous Section to one of the most traditional issues in economics, i.e. the relation between productivity and labour input (e.g. hours worked). The predictions of New Keynesians (NK) and Real-business-cycle (RBC) theorists are opposite in this respect. RBC models predict a positive correlation between labour productivity and labour input growth rates when conditioning on a technology shock. In order to replicate the almost null observed unconditional correlation between labour productivity and labour input growth rates, RBC models need to assume a negative correlation conditional to another shock (e.g. on government purchases or on preferences). NK models instead imply a negative correlation when conditioning on a technology shock and a positive correlation when conditioning on a nontechnological (monetary) shock.

When estimating a VAR on hours per worker and total labour force, Galí (1999) finds results which are consistent with a NK model. However, more recently Christiano et al. (2004) show contradictory results for the impulse response of hours per worker to a technology shock, and therefore for the derived conditional correlation. Results change depending on whether hours per worker enter the VAR in differences or in levels. In the first case, the conditional correlation is negative, while it is positive in the second case. More light is shed on this problem by two factor models applications: the first is in Giannone and Reichlin (2006), the second is presented here and is our original contribution to the applied econometric literature on nonfundamentalness.

Giannone and Reichlin (2006) augment the model by Galí (1999) using as additional variables labor productivity and labor input at sectoral level. As explained in Section 7, they find nonfundamentalness in the bivariate SVAR. Thus, they assume a factor structure for the whole augmented panel and estimate new impulse responses as explained in Section 8. Figure 5 (a) reports the estimated response of first-differenced hours, together with its 5% confidence bands, to a technology shock in the bivariate SVAR. The point estimate exhibits a significant and persistent decline in hours, the bulk of the variation taking place at impact. Giannone and Reichlin (2006) estimate the same impulse response by means of the dynamic factor model, including different numbers of common factors ($r \leq 8$) and imposing the same identification restrictions as in the SVAR. Figure 5 (b) reports the value at impact of this response together with 5% confidence bands, for different numbers of common factors (on the horizontal axis). The more factors are included in the model, i.e. the more sectoral information gets captured, the more the response is shifted upward and the contemporaneous response of hours becomes not significantly different from zero.

We use the same data as Christiano et al. (2004), i.e. labour productivity and hours worked series downloaded from the DRI Economics database (together with a measure of the civilian population over the age of 16 used to convert hours worked to per capita terms).[7] The logarithm of the series of labour input and the logarithm of GDP are indicated respectively as ℓ_t and y_t. The corresponding logarithm of labour productivity is computed as $\pi_t = y_t - \ell_t$. We augment the bidimensional SVAR with a different set of variables with respect to Giannone and Reichlin (2006). Indeed, we add to the original two variables a US macroeconomic dataset containing 60 monthly series (similar to the ones used in factor model literature, e.g. Giannone et al., 2004) and the series of quarterly GDP. We transform all monthly data in quarterly data, and the sample period chosen is from 1964:Q1 to 2006:Q4.

The criterion for determining the number of dynamic factors by Hallin and Liška (2007) suggests the presence of two common shocks in this large dataset. We report here results for

[7] DRI series codes are LBOUT, LBMN and P16, respectively.

the case $r = 10$, however they are robust for a number of static factors from $r = 6$ to $r = 18$. We use the standard identification restriction of a vanishing long-run impulse response of productivity to the nontechnological shock.

In Figure 6 we plot the roots of $\det \widehat{\mathbf{C}}(z)$ once identification is achieved.[8] When hours are in levels two roots lie inside the unit circle, while when hours are taken in differences there are no roots inside the unit circle. Indeed, our model gives results similar to the traditional SVAR when hours are in differences and different results when hours are taken in levels. The latter is precisely the case in which the dynamic factors estimated using the whole N-dimensional dataset are nonfundamental for the 2-dimensional subpanel we are interested in. The analysis of the roots shows that different results obtained by the SVARs are due to the presence of locally nonfundamental shocks. To check whether the additional series actually convey relevant information for identification, we check whether their responses to the structural shocks are heterogeneous. We plot in Figure 7 the impulse responses for two groups of variables: industrial production and price indexes. Within a group of variables the responses are similar, while between groups they differ.

Galí finds a negative correlation in growth rates between labour productivity and labour input when conditioning on a technology shock. Considering the logs of labour input and GDP, if the estimated impulse response matrix is $\widehat{\mathbf{C}}(L) = \sum_{j=0}^{K} \widehat{\mathbf{C}}_j L^j$, for a given maximum truncation lag K, then the correlation between growth rates conditional on the i-th shock is

$$\text{corr}((1-L)y_t, (1-L)\ell_t | i) = \frac{\sum_{j=0}^{K} \widehat{\mathbf{C}}_{j,1i} \widehat{\mathbf{C}}_{j,2i}}{\sqrt{\sum_{j=0}^{K} (\widehat{\mathbf{C}}_{j,1i})^2 \sum_{j=0}^{K} (\widehat{\mathbf{C}}_{j,2i})^2}}, \quad i = 1, 2.$$

Table 2 summarizes the results obtained with a SVAR and with a factor approach. In the latter case, no matter whether hours enter the model in levels or in differences, we always find a negative conditional correlation between hours and productivity when conditioning on the technology shock (-0.35 and -0.53 in the two cases, respectively). In the case of a bivariate SVAR, the conditional correlation between hours and productivity when conditioning on a technology shock is positive (0.21) if hours enter in levels while is negative (-0.87) if hours enter in differences, consistently with the results by Christiano et al. (2004). The conditional correlation between hours and productivity conditional on a non-technological shock is estimated to be positive in a factor model (0.46 and 0.47 when hours are in levels and in differences, respectively), while it is estimated to be negative (-0.67) in a SVAR with hours in levels and positive (0.48) in a SVAR with hours in differences. Our results are therefore in favour of the New-Keynesian interpretation by Galí who finds a negative correlation between labour productivity and hours worked induced by technological progress. This negative correlation is counterbalanced by a positive correlation induced by a monetary shock, resulting into almost uncorrelated series (the observed sample unconditional correlation is -0.03 in our dataset).

[8]Notice that identification is not a necessary step to determine if the roots are inside the unit circle. Indeed if a representation is nonfundamental before applying an orthogonal transformation, it is still nonfundamental afterwards.

10 Concluding remarks and further research

We have described examples of meaningful economic models which generate, endogenously or exogenously, nonfundamental MA models. Such models can arise in very simple models with rational expectations, or with heterogeneous information, but also in more complex cases as in DSGE models. Since a VAR white noise is fundamental by construction, the nonfundamental shocks cannot be identified by estimating and identifying a SVAR. In other words, SVARs do not allow to recover the structural shocks in all those cases in which the structural shocks are nonfundamental for the small set of variables considered. This is likely the case when the economic agent's information space is larger than the econometrician's one.

When there is nonfundamentalness, SVARs are not useful for discriminating among competing economic models. We have recalled the standard conditions for fundamentalness and an alternative condition by Fernández-Villaverde et al. (2007), which can be used as a test to check whether a given DSGE model produces a fundamental representation and therefore the impulse responses of the associated VAR are consistent with the theoretical impulse responses. If this is not the case, one can resort to dynamic factor models. As shown in Forni et al. (2009), if we represent data with a factor structure nonfundamentalness can be made non generic by exploiting heterogenous cross-sectional information.

To show the importance of nonfundamental models in practice, we considered the response of labour input to a technology shock. Results from a bivariate SVAR depend on whether hours worked are taken in differences or not, while what we find by means of a factor model is that a technology shock generates a negative correlation between hours per worker and productivity growth rates, independently of the transformation used. Indeed, when hours are taken in levels we find nonfundamental shocks for the two-dimensional subsystem, which cannot be recovered with a SVAR.

An alternative strategy to factor models is to generate nonfundamental models from the only fundamental one estimated with a SVAR. This can be done by means of Blaschke matrices, which are filters capable to flip the roots of a fundamental representation inside the unit circle. Lippi and Reichlin (1993) have applied this procedure, but much can still be done as far as the search for nonfundamental representations is concerned. For instance, we could try to identify a correspondence between the roots of a given MA of an economic model and the associated impulse responses by exploring whole regions of the parameter space. The same method would allow us to find theoretical impulse responses which may derive also from nonfundamental representations and are consistent both with the data and with the structural model.

Moreover, it would be interesting to discuss the identification of structural shocks within the framework of monetary transmission mechanism (MTM) as well as for the estimation of the effects of fiscal policy shocks. Monetary policy and fiscal policy are fields in which, in general, nonfundamentalness plays an important role since economic models can produce nonfundamental econometric representations insofar as agents are characterized by rational expectations. Indeed, being agents forward-looking, they are likely to possess a wider information set than the policy maker (i.e. the econometrician) and anticipate the effects of any foreseen future intervention by the Central Bank or the Government, let alone the large amount of information available to the agents but not included in a low-dimensional SVAR. As a consequence, an unexpected intervention by the Central Bank might not necessarily coincide with the fundamental monetary shock identified in a VAR, and this might yield misleading results on the effects of tax policies. The intuition of the role of nonfundamental shocks in the analysis of the MTM dates back to Sims (1992), who explained the price puzzle - i.e. an increase in the price level following a contractionary shock - by pointing out that the Central Bank might have information on future inflation, which the econometrician estimating a VAR does not have. As far as we know, very few studies are available where nonfundamental representations of the MTM are investigated. By adopting such an approach, Klaeffing (2003) explains some puzzles concerning the effects of a monetary shock on output while Giannone et al. (2008), identifying nonfundamental shocks in the pre-Volcker period, show that the cause of the Great Moderation is not a decline in the volatility of the shocks. Beyer and Farmer (2007a,b) show via simulations that determinate and indeterminate models may be observationally equivalent. Indeed, nonfundamentalness is precisely the counterpart of indeterminacy of equilibria in the economic model since structural shocks can be nonfundamental (i.e. sunspot) if equilibria are indeterminate. On fiscal policy, a recent work by Leeper et al. (2008) provides evidence of fiscal foresight and shows that this intrinsic feature of the tax policy process implies nonfundamentalness in the econometric model.

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A The search for nonfundamental representations

A.1 MA models and Blaschke matrices

Nonfundamental representations can be generated by means of Blaschke matrices, which are defined as follows (see Lippi and Reichlin (1994) for additional details)

Definition A-1 (Blaschke Matrix) *A complex-valued matrix* $\mathbf{B}(z)$ *is a Blaschke matrix if:*

1. *it has no poles inside the unit circle;*
2. $\mathbf{B}(z)^{-1} = \overline{\mathbf{B}'}(z^{-1})$.[9]

Whenever we apply a Blaschke matrix to a fundamental MA process we get a nonfundamental representation defined as

$$\mathbf{x}_t = \mathbf{C}(L)\mathbf{B}(L)\mathbf{B}(L)^{-1}\mathbf{u}_t = \mathbf{D}(L)\mathbf{v}_t, \quad t \in \mathbb{Z}, \tag{A-1}$$

where starting from an invertible $\mathbf{C}(L)$ with fundamental shocks $\mathbf{u}_t$, we get a two-sided filter $\mathbf{D}(L)$ with nonfundamental shocks $\mathbf{v}_t$. The main property we need is that if $\mathbf{u}_t$ is an orthonormal white noise then $\mathbf{v}_t = \mathbf{B}(L)\mathbf{u}_t$ is an orthonormal white noise if and only if $\mathbf{B}(L)$ is a Blaschke matrix. This ensures that, also nonfundamental shocks, are uncorrelated, which is a necessary condition in all econometric models. Thus, (A-1) together with usual identification restrictions is still a valid structural model with new impulse responses that are not recoverable with an ordinary VAR.

As examples of Blaschke matrices we have the orthogonal matrices and the matrices with a Blaschke factor as one of the entries. A generic Blaschke matrix can be always written as the product of these two.

Theorem A-1 *Let* $\mathbf{B}(z)$ *be an* $N \times N$ *Blaschke matrix then* $\exists\ m \in \mathbb{N}$ *and* $\exists\ \alpha_i \in \mathbb{C}$ *s.t.* $|\alpha_i| < 1$ *for* $i = 1, \dots, m$ *and*

$$\mathbf{B}(z) = \prod_{i=1}^{m} \mathbf{K}(\alpha_i, L)\, \mathbf{R}_i = \prod_{i=1}^{m} \begin{pmatrix} \frac{z-\alpha_i}{1-\overline{\alpha}_i z} & 0 \\ 0 & I_{N-1} \end{pmatrix} \mathbf{R}_i, \quad z \in \mathbb{C}, \tag{A-2}$$

where $\mathbf{R}_i \overline{\mathbf{R}}_i' = \mathbf{I}_N$.

Notice that $\mathbf{B}(z)$ has poles in $(\overline{\alpha}_i)^{-1}$, i.e. outside the unit circle as required by definition A-1.

With reference to (A-1), given a fundamental representation $\mathbf{x}_t = \mathbf{C}(L)\mathbf{u}_t$, let us consider the zeroes of $\det \mathbf{C}(z)$, which by definition are all outside the unit circle, and call them γ_i. We can build a nonfundamental representation just by applying a Blaschke matrix $\mathbf{B}(L)$ to $\mathbf{C}(L)$ with $\alpha_i = (\overline{\gamma_i})^{-1}$ for $i = 1, \dots, m$ and $1 \le m \le N$. Theorem 1 tells us that $\mathbf{B}(L)$ is taking zeroes of $\mathbf{C}(L)$, that are outside the unit circle ($|\gamma_i| > 1$), into zeroes of $\mathbf{D}(L)$ which are inside the unit circle ($|\alpha_i| = |(\overline{\gamma_i})^{-1}| < 1$).

Finally, notice that $\mathbf{x}_t = \mathbf{C}(L)\mathbf{B}(L)\mathbf{v}_t$, therefore $\mathbf{B}(L)^{-1}\mathbf{C}(L)^{-1}\mathbf{x}_t = \mathbf{v}_t$, but, although $\mathbf{C}(L)$ is invertible in the past (i.e. is fundamental) by construction, the inverse of a Blaschke matrix requires the use of L^{-1} (the forward operator), therefore it is impossible to recover $\mathbf{v}_t$ only from the past of $\mathbf{x}_t$: this is nonfundamentalness.

[9] With the bar we indicate the matrix obtained by taking complex conjugate coefficients

A.2 ARMA models

We now move to ARMA representations $\mathbf{M}(L)\mathbf{x}_t = \mathbf{C}(L)\mathbf{u}_t$, where $\det \mathbf{M}(z)$ has no zeroes inside the unit circle in order to guarantee stationarity and causality for the AR component. The ARMA representation is fundamental if its MA component, $\mathbf{C}(L)\mathbf{u}_t$, is fundamental. Lippi and Reichlin (1994) look for different ARMA specifications where, while the AR component is completely identified, the MA component is identified up to a Blaschke matrix transformation. They point out how many examples of intertemporal maximization under rational expectations produce indeed such a situation, as discussed in Section 5. If $\mathbf{C}(L)$ is fundamental then its determinant has all $h \leq N$ roots α_i outside the unit circle, hence we can build nonfundamental representations $\mathbf{D}(L)$ just by moving one or more roots of $\det \mathbf{C}(z)$ from outside to inside the unit circle by means of a Blaschke matrix. In order to do so, first define the subset $\Omega \in \mathbb{R}^h$ such that $\Omega = \{\boldsymbol{\omega} = (\omega_1 \dots \omega_h) \, \text{s.t.} \, \omega_i = \pm 1\}$. We have the following theorem:

Theorem A-2 *For any possible $\omega \in \Omega$ there exist representations $\mathbf{M}(L)\mathbf{x}_t = \mathbf{P}(L)\mathbf{v}_t$ such that $\det \mathbf{P}(z)$ has h roots β_i defined as*

$$\begin{aligned} \beta_i &= \alpha_i \qquad \text{if} \quad \omega_i = 1\,, \\ \beta_i &= (\overline{\alpha_i})^{-1} \quad \text{if} \quad \omega_i = -1 \quad i = 1, \dots, h. \end{aligned}$$

Moreover, if $\mathbf{P}(L)$ and $\mathbf{Q}(L)$ correspond to the same ω, then $\mathbf{P}(L) = \mathbf{K}\mathbf{Q}(L)$ with $\mathbf{K}$ orthogonal, i.e. the two representations are unique up to a rotation.

Notice that if at least one of the elements of $\boldsymbol{\omega}$ is -1 then $\mathbf{P}(L)$ will be a nonfundamental representation. All the nonfundamental representations obtained in this way are called basic. They come from an ARMA just by transforming the MA part while leaving untouched the AR part. Moreover, if we start from an ARMA(p,q) then all its basic representations are ARMA(p,q). Non-basic representations are obtained by multiplying the MA component $\mathbf{C}(L)$ by an arbitrary Blaschke matrix. By doing so we increase the order of the MA and AR polynomials and if γ is a nonfundamental root of the MA, then $(\overline{\gamma})^{-1}$ is a root of the AR part. Both common sense and literature suggest that this latter case is not likely to occur, thus it makes sense to search only for basic nonfundamental representations.

A.3 VAR models

In general we always start from an estimated VAR, and, once inverted, we get an MA representation that by definition has fundamental shocks. However, from the latter we can always get nonfundamental shocks that generate the impulse responses of our alternative theoretical model. This is the procedure followed by Lippi and Reichlin (1993) to generate impulse responses that represent technological diffusion under learning-by-doing dynamics. Such method is well explained by Lippi and Reichlin (1994). If the true (i.e. implied by economic theory) fundamental MA model $\mathbf{x}_t = \mathbf{C}(L)\mathbf{u}_t$ were known then all its nonfundamental counterparts would easily be recovered just by applying a Blaschke matrix as in (A-1). However, from an estimated VAR, $\mathbf{A}(L)\mathbf{x}_t = \mathbf{u}_t$, we can only get the approximate ARMA

$$(\det \mathbf{A}(L))\, \mathbf{x}_t = \mathbf{A}_{ad}(L)\mathbf{u}_t, \quad t \in \mathbb{Z}.$$

Its associated approximate MA representation is $\mathbf{x}_t = \mathbf{T}(L)\mathbf{u}_t$ with $\mathbf{T}(L) = (\det \mathbf{A}(L))^{-1}\, \mathbf{A}_{ad}(L)$. We have approximations because these are all finite order representations, although in theory they should have an infinite MA part or, viceversa, if the true MA were of finite order, then we should estimate an infinite VAR.

As an example, Lippi and Reichlin (1994) consider the following two-dimensional MA:

$$\mathbf{x}_t = \mathbf{C}(L)\mathbf{v}_t = (\mathbf{I} - \mathbf{C}L)\mathbf{u}_t \quad t \in \mathbb{Z}.$$

They assume that $\det(\mathbf{I} - \mathbf{C}z)$ has two roots α_1 and α_2, which by fundamentalness are both outside the unit circle ($|\alpha_i| > 1$). The VAR that we estimate is only the order p approximation

$$\mathbf{A}(L) = \mathbf{I} + \sum_{k=1}^{p} \mathbf{C}^k L^k \simeq (\mathbf{I} - \mathbf{C}L)^{-1}.$$

It is possible to show that the $2p$ complex roots of $\det \mathbf{A}(z)$ are

$$\alpha_i \exp\left(k\frac{2\pi i}{p+1}\right) \quad \text{for } i = 1, 2 \text{ and } k = 1, \dots, p.$$

Therefore, the roots of the VAR are all on circles of radius $|\alpha_i| > 1$. If the roots of the MA are complex we have only one circle of roots, if instead they are real we have two circles. Here we consider the case of two complex conjugate roots $\alpha_1 = \overline{\alpha}_2$.

Actually, we are able only to get an estimate of $\mathbf{A}(L)$, thus we cannot estimate directly the roots of $\mathbf{C}(L)$. But we can determine the radius ρ of the circle where the roots of $\mathbf{A}(L)$ lie. For every complex β such that $|\beta| = \rho$, we proceed as though β were a root of $\mathbf{T}(L)$, which is only an approximation of $\mathbf{C}(L)$. We therefore apply theorems 1 and 2 by multiplying $\mathbf{T}(L)$, which is by construction a fundamental representation, by a Blaschke matrix in order to obtain a nonfundamental representation. First, we look for a rotation $\mathbf{R}$ such that $\mathbf{T}(z)\mathbf{R}$ has in its first column the factor $(z - \beta)$. $\mathbf{R}$ has to satisfy

$$[\mathbf{T}(z)\mathbf{R}]\mathbf{e}_1 = (z - \beta)\mathbf{e}_1, \quad z \in \mathbb{C}, \tag{A-3}$$

where $\mathbf{e}_1 = (1 \ \ 0)'$. Note that (A-3) is a condition only on the first column of $\mathbf{R}$, while the second column is obtained just by using the orthogonality condition: $\mathbf{R}\overline{\mathbf{R}}' = \mathbf{I}$. If the system were N-dimensional we would determine unambiguously only the first column of $\mathbf{R}$ while no rule exists for fixing all other columns besides the orthogonality condition. After rotating $\mathbf{T}(z)$, we can move the root, that now is in the first column, from β to $(\overline{\beta})^{-1}$ with

$$\mathbf{K}\left((\overline{\beta})^{-1}, L\right) = \begin{bmatrix} \frac{z-(\overline{\beta})^{-1}}{1-\beta^{-1}z} & 0 \\ 0 & 1 \end{bmatrix}. \tag{A-4}$$

We thus obtain a nonfundamental representation

$$\mathbf{x}_t = \mathbf{T}(L)\mathbf{B}(L)\mathbf{B}(L)^{-1}\mathbf{u}_t, \quad t \in \mathbb{Z} \tag{A-5}$$

where $\mathbf{B}(L) = \mathbf{R}\mathbf{K}\left((\overline{\beta})^{-1}, L\right)$. Actually, since we know only ρ, we need to repeat this procedure n times in order to explore all the circle of roots of $\mathbf{A}(L)$. We choose $\beta = \rho\exp(ik\theta)$ with $\theta = \pi/n$, and $k = 1, \dots, n-1$, n being the number of roots. Notice however that since we consider all β on the circle we are taking in account not only the roots of $\mathbf{C}(L)$ but also other values, therefore we are looking also for non-basic representations. This in turn implies that no uniqueness result as in Theorem A-2 holds in this case. Finally, we can study the impulse responses of the nonfundamental innovations, see if some of them are economically sensible and possibly assess differences with the fundamental impulse responses $\mathbf{T}(L)$. Although this is only an approximate procedure, it has already delivered promising results in Lippi and Reichlin (1994).

Table 1: Fraction of the forecast-error variance attributed to the permanent shock.

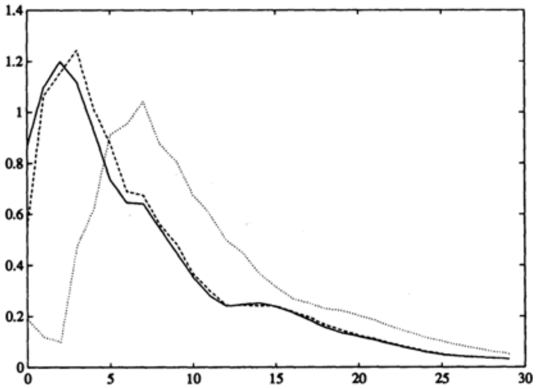
Horizon	**Dynamic Factor Model**			**SVAR**		
(quarters)	Output	Consumption	Investment	Output	Consumption	Investment
1	0.37	0.30	0.07	0.45	0.88	0.12
4	0.37	0.30	0.07	0.45	0.88	0.12
8	0.78	0.87	0.72	0.68	0.83	0.40
12	0.86	0.90	0.80	0.73	0.83	0.43
16	0.89	0.91	0.83	0.77	0.85	0.44
20	0.91	0.92	0.86	0.79	0.87	0.46

Source: Forni et al. (2009).

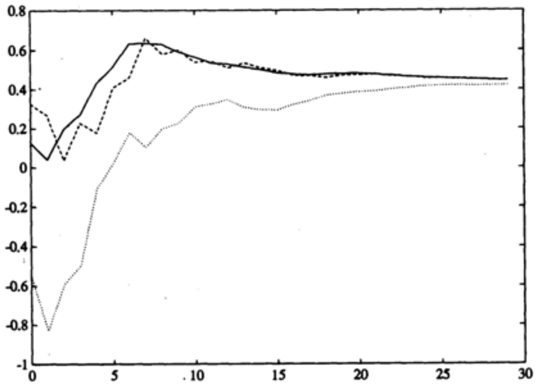
Table 2: Correlation of hours worked and productivity conditional on the shocks.

	conditional on technology shock	**conditional on non-technology shock**
VAR - hours in levels	0.21	-0.67
VAR - hours in differences	-0.87	0.48
DFM - hours in levels	-0.35	0.46
DFM - hours in differences	-0.53	0.47

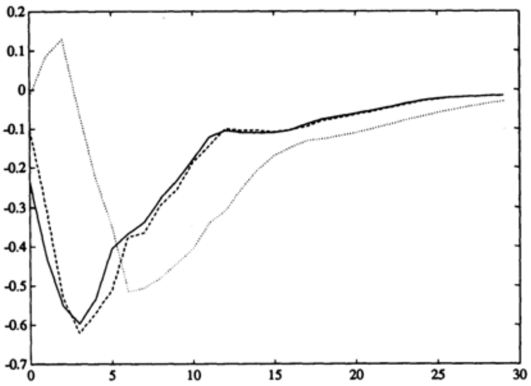
VAR = results obtaind when estimating a SVAR on the two-dimensional panel. DFM = results obtained when estimating a dynamic factor model on the large N-dimensional panel.



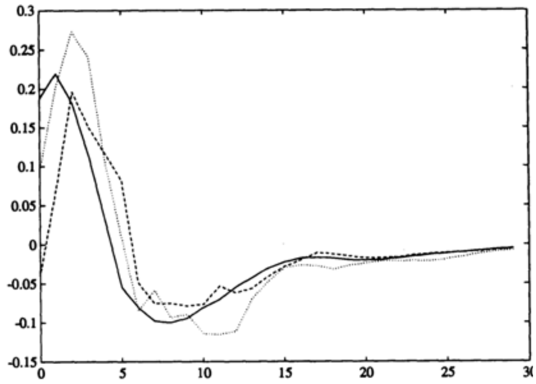
(a) Output response to demand.



(b) Output response to supply.



(c) Unemployment response to demand.



(d) Unemployment response to supply.

Figure 1: Solid line: impulse response to fundamental innovations; dashed and dotted lines: impulse responses to nonfundamental innovations. *Source*: Lippi and Reichlin (1993).

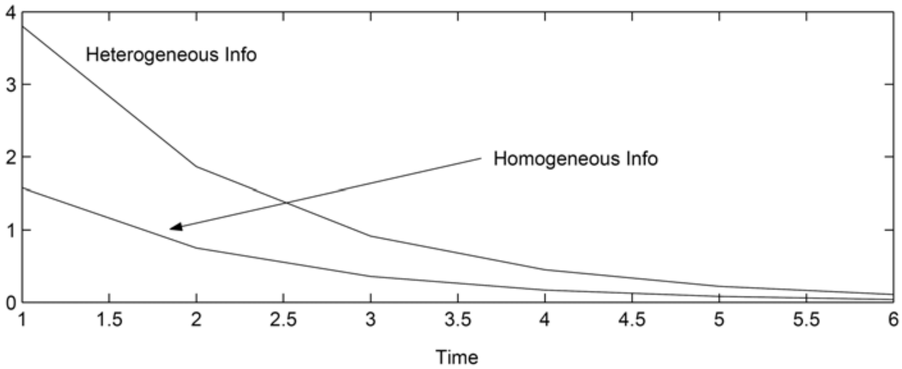


Figure 2: Asset price response to a shock in market fundamentals when the MA representation is fundamental (homogeneous information case) and when it is nonfundamental (heterogeneous information case). *Source*: Kasa et al. (2006).

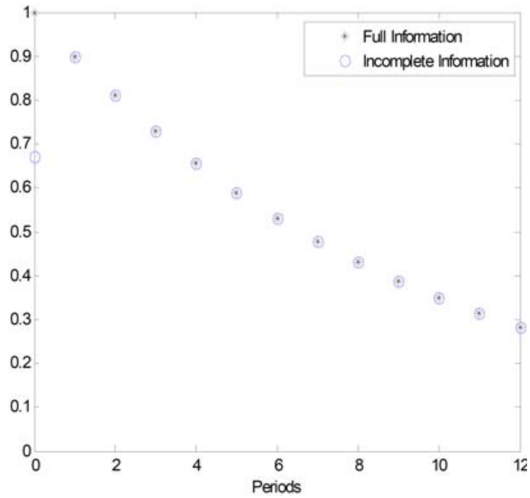


Figure 3: Aggregate price response to a shock in aggregate productivity when the MA representation is fundamental (full information case) and when it is nonfundamental (incomplete information case). *Source*: Rondina (2008).

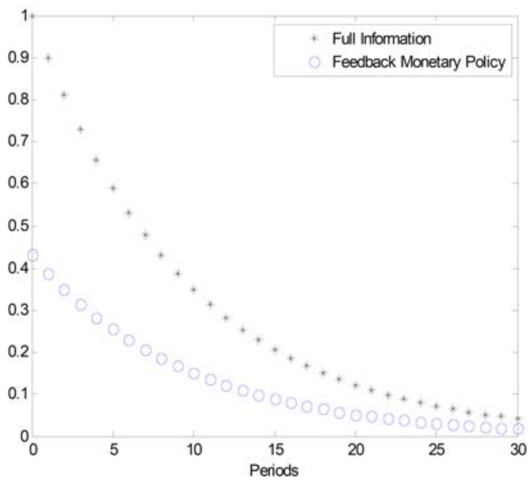
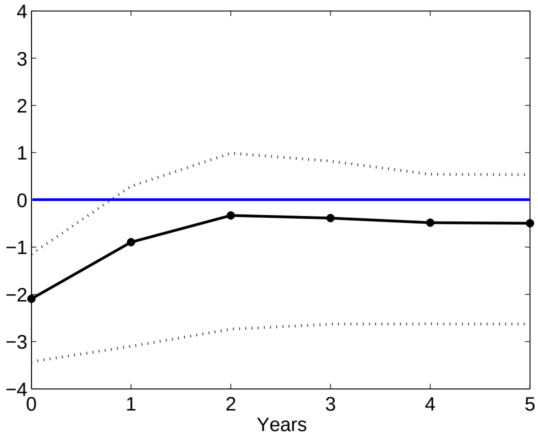
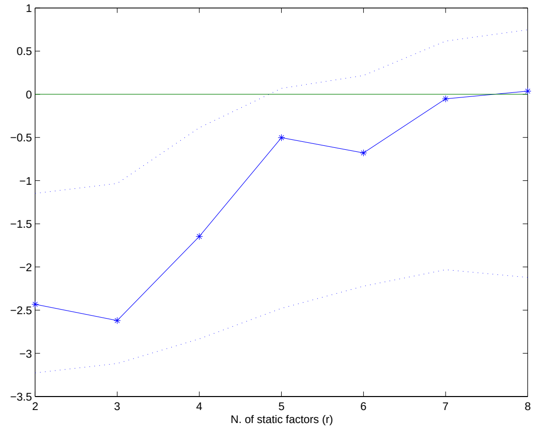


Figure 4: Impulse response of the equilibrium aggregate price in the homogeneous-full information case and when a control is implemented to turn the representation from nonfundamental to fundamental. *Source*: Rondina (2008).

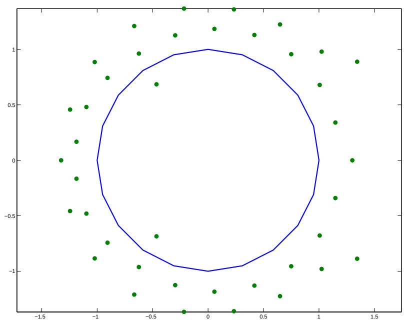


(a) Response from a bivariate VAR.

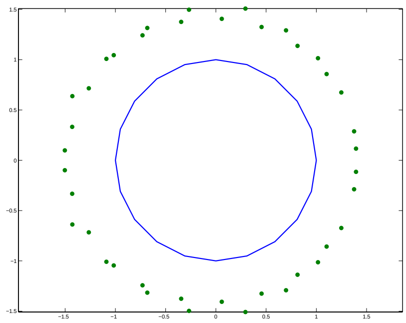


(b) Response at impact from a large scale system.

Figure 5: Estimated impulse response of hours to a technology shock. *Source*: Giannone and Reichlin (2006).

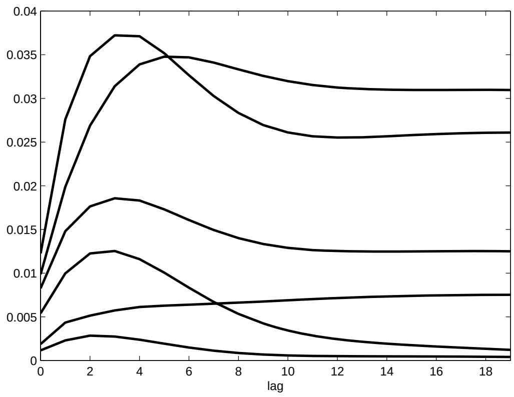


(a) Hours in levels.

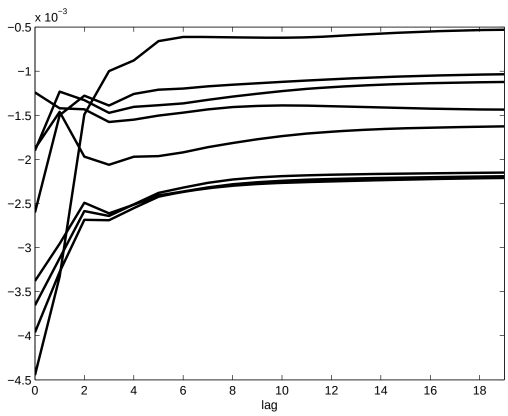


(b) Hours in differences.

Figure 6: Evidence of nonfundamental representations.



(a) Industrial production response to a technology shock.



(b) Prices response to a non-technology shock.

Figure 7: Evidence of dynamic heterogeneity in the impulse responses.