Notes on Series

Matteo Barigozzi*

Basics

Definition 1 *Given the sequence* $\{a_n\} \subset \mathbb{R}^k$*, we associate to it the sequence of partial sums* $\{s_n\}$ *where the generic term is defined as*

$$s_N = \sum_{n=1}^{N} a_n$$

so that

$$\{s_n\} = (a_1, a_1 + a_2, \ldots) \stackrel{def}{=} \sum_{n=1}^{\infty} a_n$$

The series is the sequence of partial sums and if $\{s_n\}$ *converges then*

$$\lim_{n\to\infty} s_n = \sum_{n=1}^{\infty} a_n < \infty$$

The series is a limit NOT a sum.

Theorem 1 $\sum a_n$ *converges if and only if*

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \ s.t. \ \ m \geq n \geq N \ \Rightarrow \left| \sum_{k=m}^{n} a_k \right| < \varepsilon$$

*matteo.barigozzi@gmail.com

As an intuition of the proof notice that if $\lim s_n = s$ then

$$\forall \varepsilon > 0 \;\; \exists N \in \mathbb{N} \;\text{ s.t. }\; n \geq N \;\Rightarrow\; |s - s_n| = \left| \sum_{k=n+1}^{\infty} a_k \right| < \varepsilon$$

Now consider the previous theorem with $m = n$, we have a necessary condition for convergence.

Theorem 2 (Necessary Condition)

$$\sum_{n=1}^{\infty} a_n \textit{ converges } \Rightarrow \lim_{n\to\infty} a_n = 0$$

As an intuition of the proof notice that $a_n = s_n - s_{n-1}$ then

$$\{s_n\} \text{ converges } \Rightarrow \lim_{n\to\infty} s_n = \lim_{n\to\infty} s_{n-1} = s \Rightarrow \lim_{n\to\infty} a_n = 0$$

It is only a necessary condition, indeed for example $\sum \frac{1}{n}$ diverges although $\frac{1}{n} \to 0$.

Theorem 3 (Necessary and Sufficient Condition) *If $a_n \geq 0$ for any $n \in \mathbb{N}$ then*

$$\sum_{n=1}^{\infty} a_n \textit{ converges } \Leftrightarrow \{s_n\} \textit{ is bounded}$$

Theorem 4 (Comparison Criterion) *The following two criteria hold*

1. *If $|a_n| \leq c_n$ for some N such that $n \geq N$ and $\sum c_n$ converges then $\sum a_n$ converges;*
2. *If $a_n \geq d_n \geq 0$ for some N such that $n \geq N$ and $\sum d_n$ diverges then $\sum a_n$ diverges.*

Note that the criterion applies only to series with nonnegative terms $a_n \geq 0$.

Theorem 5 *The absolute convergence implies convergence (but not the viceversa):*

$$\sum_{n=1}^{\infty} |a_n| \textit{ converges } \Rightarrow \sum_{n=1}^{\infty} a_n \textit{ converges}$$

This can be proved using the triangular inequality

$$\sum_{n=1}^{\infty} |a_n| > \left| \sum_{n=1}^{\infty} a_n \right|$$

and then using the comparison criterion.

The geometric series

The geometric series of ratio x is defined as

$$\sum_{k=0}^{\infty} x^k \qquad x \geq 0$$

To study its convergence we have to compute the limit of the sequence of partial sums

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=0}^{n} x^k$$

Now multiply by $(1-x)$ and we get

$$(1-x)s_n = (1-x)(1+x+\ldots+x^n) = (1+x+\ldots+x^n) - (x+x^2+\ldots+x^{n+1}) = 1-x^{n+1}$$

Therefore

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{1-x^{n+1}}{1-x} = \begin{cases} \frac{1}{1-x} & \text{if} \quad 0 \leq x < 1 \\ \\ \infty & \text{if} \qquad x \geq 1 \end{cases}$$

If $x < 0$ then $\sum |x^k|$ converges if $|x| < 1$ and then $\sum x^k$ converges if $|x| < 1$. Finally, notice that

$$\sum_{k=m}^{\infty} x^k = \sum_{k=0}^{\infty} x^{k+m} = x^m \sum_{k=0}^{\infty} x^k$$

The harmonic series

The harmonic series is defined as

$$\sum_{k=0}^{\infty} \frac{1}{k}^p$$

The series converges if $p > 1$ and diverges if $p \leq 1$.

p=1

The series diverges, indeed, although $\lim \frac{1}{n} = 0$, we have

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n} &= 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{>\frac{1}{2}} + \underbrace{\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}}_{>\frac{1}{2}} + \dots \\ &> 1 + \frac{1}{2} + \frac{1}{2} + \dots \\ &= \sum_{n=1}^{\infty} n = \infty\end{aligned}$$

p= 0

The series diverges since $\lim \frac{1}{n^p} = 1$.

p< 0

The series diverges since $\lim \frac{1}{n^p} = \infty$.

p=2

The series coverges, indeed

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} < 1 + \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 + \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{(n+1)} \quad (*)$$

The generic term of the sequence of partial sums for last series is

$$s_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \dots + \left(\frac{1}{n} - \frac{1}{(n+1)}\right) = 1 - \frac{1}{n+1}$$

Therefore $\lim s_n = 1$ so the series $(*)$ converges and thus $\sum \frac{1}{n^2}$ converges.

p>2

The series converges because $\frac{1}{n^p} < \frac{1}{n^2}$.

1<p<2

The series converges, indeed

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n^p} &= 1+\underbrace{\frac{1}{2^p}+\frac{1}{3^p}}_{2 terms}+\underbrace{\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}}_{2^2 terms}+\ldots+\underbrace{\frac{1}{(2^n)^p}+\frac{1}{(2^n+1)^p}+\ldots+\frac{1}{(2^n+2^n-1)^p}}_{2^n terms}+\ldots\\ &< 1+2\frac{1}{2^p}+\ldots+2^n\frac{1}{(2^n)^p}+\ldots\\ &= \sum_{n=1}^{\infty}\frac{1}{(2^{p-1})^n}\end{aligned}$$

The last series is a geometric series of ratio $\frac{1}{2^{p-1}}<1$ thus it converges, therefore $\sum\frac{1}{n^p}$ converges.

The number e

The number e is defined as

$$e=\sum_{n=0}^{\infty}\frac{1}{n!}=2.7183$$

The generic term of the sequence of partial sums for this series is

$$s_n=1+1+\frac{1}{2}+\frac{1}{3!}+\ldots+\frac{1}{n!}<1+1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}}<3$$

Thus $e<3$ moreover the rate of convergence is determined by

$$e-s_n<\frac{1}{n!n}$$

Theorem 6 *The number e can be defined also as*

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$$

Other criteria for convergence

Theorem 7 (Root Test) *Given the series* $\sum a_n$ *define*

$$\alpha = \lim_{n\to\infty} \sup \sqrt[n]{|a_n|} \Rightarrow \begin{cases} \textit{if} \quad \alpha < 1 & \textit{the series converges} \\ \textit{if} \quad \alpha > 1 & \textit{the series diverges} \\ \textit{if} \quad \alpha = 1 & \textit{we don't know} \end{cases}$$

Intuition of the proof. If $\alpha < 1$ then $\exists\, \beta$ such that $\alpha < \beta < 1$ and $\exists\, N$ such that $\forall\, n \geq N$ we have that $|a_n| < \beta^n$. Thus the series $\sum |a_n|$ is bounded by a geometric series of ratio $\beta < 1$ and we have absolute convergence and thus convergence of $\sum a_n$.
If $\alpha > 1$ then $\exists\, n_k$ such that $|a_{n_k}| \to \alpha^{n_k} > 1$ therefore $\exists\, N$ such that for $n_k \geq N$ $|a_{n_k}| > 1$ and the necessary condition for convergence of the seires doesn't hold. Finally, consider $\sum \frac{1}{n}$ and $\sum \frac{1}{n^2}$ for both $\alpha = 1$ but the first diverges, while the second converges.

Example Consider the series

$$\sum_{n=1}^{\infty} \left(\frac{n}{n+1} \right)^{n^2}$$

Then we apply the root test and compute α by using the de L'Hôpital rule:

$$\alpha = \lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{n+1} \right)^{n^2}} = \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^{n}$$

$$\log \alpha = \lim_{n\to\infty} n \log \left(\frac{n}{n+1} \right) \overset{H}{=} \lim_{n\to\infty} -\frac{n^2}{n(n+1)} = -1$$

Therefore $\alpha = e^{-1} < 1$ and the series converges.

Theorem 8 (Ratio Test) *Given the series* $\sum a_n$ *define*

$$\beta = \lim_{n\to\infty} \sup \left| \frac{a_{n+1}}{a_n} \right| \Rightarrow \begin{cases} \textit{if} \quad \beta < 1 & \textit{the series converges} \\ \textit{if} \quad \beta \geq 1 & \textit{the series diverges} \end{cases}$$

Example The series $\sum \frac{1}{n!}$ has $\beta = 0$ therefore converges.

Example The series $\sum \frac{2^n}{n}$ has $\beta = 2$ therefore diverges.

Theorem 9 *Given the series* $\sum a_n$ *and* $\sum b_n$ *such that* $a_n \geq 0$ *and* $b_n \geq 0$*, then if* $\sum a_n$ *converges and*

$$\frac{b_{n+1}}{b_n} < \frac{a_{n+1}}{a_n}$$

it implies that $\sum b_n$ *converges.*

In practice define

$$\gamma = \lim_{n\to\infty} \frac{b_n}{a_n} \Rightarrow \begin{cases} \text{if} \quad \gamma \neq 0 \quad \text{both series converge or diverge} \\ \text{if} \quad \gamma = 0 \quad \text{if} \quad \sum a_n \quad \text{converges} \Rightarrow \sum b_n \quad \text{converges} \\ \text{if} \quad \gamma = \infty \quad \text{if} \quad \sum a_n \quad \text{diverges} \Rightarrow \sum b_n \quad \text{diverges} \end{cases}$$

Example Consider the series

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \frac{4n^2 + 5n - 2}{n(n^2+1)^{3/2}}$$

Then

$$c_n < \frac{4n^2 + 5n - 2}{n^4} = a_n$$

and consider the series $\sum b_n = \sum \frac{1}{n^2}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 4$$

Since $\sum b_n$ converges then also $\sum a_n$ converges and therefore $\sum c_n$ converges.

Main Reference

Rudin, W. *Principles of Mathematical Analysis* McGraw-Hill, Inc. 1976