

# Notes on the Riemann-Stieltjes Integral

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## The Riemann Integral

**Definition 1 (Partition)** Given  $[a, b] \subset \mathbb{R}$  a partition  $P$  of  $[a, b]$  is defined as

$$P = \{x_0 \dots x_n : a = x_0 \leq x_1 \leq \dots \leq x_{n-1} \leq x_n = b\}$$

And we define  $\Delta x_i = x_i - x_{i-1}$  for  $i = 1, \dots, n$

Now consider a function  $f : [a, b] \rightarrow \mathbb{R}$  and bounded, i.e.  $\inf_{[a,b]} f < \infty$  and  $\sup_{[a,b]} f < \infty$ .

Then for any partition  $P$  of  $[a, b]$  we define

$$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) \quad \text{and} \quad m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$$

and we can define the Riemann sums as

$$U(P, f) = \sum_{i=1}^n M_i \Delta x_i \quad \text{and} \quad L(P, f) = \sum_{i=1}^n m_i \Delta x_i$$

Notice that  $L(P, f) \leq U(P, f)$  since  $m_i \leq M_i$  for any  $i$ .

**Definition 2 (Riemann Integral)** Define the upper and lower integrals as

$$\inf_P U(P, f) = \overline{\int_a^b} f dx \quad \text{and} \quad \sup_P L(P, f) = \underline{\int_a^b} f dx$$

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For any function  $f$  bounded they always exist. If they are equal we say that the function  $f$  is Riemann integrable on  $[a, b]$ , i.e.  $f \in \mathcal{R}$  and the integral is defined as

$$\int_a^b f dx = \overline{\int_a^b f dx} = \underline{\int_a^b f dx}$$

It is the area below  $f$  in  $[a, b]$  (note that it is a signed area therefore it can be negative).

**Example** Take the Dirichlet function

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

Then, for any partition  $P$ ,  $\min f = 0$  and  $\max f = 1$  therefore  $L(P, f) = 0$  and  $U(P, f) = b - a$ .

The integral does not exist.

## The Riemann-Stieltjes Integral

Consider a function  $\alpha$  monotonically increasing on  $[a, b]$  (then it is also bounded) and, given a partition  $P$ , define  $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$ , then  $\Delta\alpha_i \geq 0$  for any  $i$ . For any  $f : [a, b] \rightarrow \mathbb{R}$  bounded and for any  $P$  we define the Riemann sums

$$U(P, f, \alpha) = \sum_{i=1}^n M_i \Delta\alpha_i \quad \text{and} \quad L(P, f, \alpha) = \sum_{i=1}^n m_i \Delta\alpha_i$$

**Definition 3 (Riemann-Stieltjes Integral)** Define the upper and lower integrals as

$$\inf_P U(P, f, \alpha) = \overline{\int_a^b f d\alpha} \quad \text{and} \quad \sup_P L(P, f, \alpha) = \underline{\int_a^b f d\alpha}$$

For any function  $f$  bounded they always exist. If they are equal we say that the function  $f$  is Riemann-Stieltjes integrable on  $[a, b]$ , i.e.  $f \in \mathcal{R}(\alpha)$  and the integral is defined as

$$\int_a^b f d\alpha = \overline{\int_a^b f d\alpha} = \underline{\int_a^b f d\alpha}$$

Hereafter we always make the two assumptions:

$$f : [a, b] \rightarrow \mathbb{R} \text{ bounded and } \alpha : [a, b] \rightarrow \mathbb{R} \text{ monotonically increasing}$$

**Definition 4 (Refinement of a Partition)**  $P^*$  is a refinement of  $P$  if  $P \subset P^*$ . Moreover  $P^*$  is a common refinement of  $P_1$  and  $P_2$  if  $P_1 \cup P_2 \subset P^*$ .

**Theorem 1** If  $P^*$  is a refinement of  $P$  then

$$L(P, f, \alpha) \leq L(P^*, f, \alpha) \quad \text{and} \quad U(P, f, \alpha) \geq U(P^*, f, \alpha)$$

In general for any  $P_1$  and  $P_2$  we have that

$$L(P_1, f, \alpha) \leq U(P_2, f, \alpha)$$

$$L(P_1, f, \alpha) \leq L(P_1 \cup P_2, f, \alpha)$$

$$U(P_2, f, \alpha) \geq U(P_1 \cup P_2, f, \alpha)$$

**Theorem 2**

$$\overline{\int_a^b} f d\alpha \geq \underline{\int_a^b} f d\alpha$$

**Theorem 3 (Existence)**  $f \in \mathcal{R}(\alpha)$  on  $[a, b]$  if and only if

$$\forall \varepsilon > 0 \exists P \text{ s.t. } U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^n (M_i - m_i) \Delta\alpha_i < \varepsilon$$

This is equivalent to say that given a  $t_i \in [x_{i-1}, x_i]$

$$\left| \sum_{i=1}^n f(t_i) \Delta\alpha_i - \int_a^b f d\alpha \right| < \varepsilon$$

**Theorem 4 (Necessary Condition)** If  $f$  is continuous in  $[a, b]$  then  $f \in \mathcal{R}(\alpha)$  in  $[a, b]$ .

**Proof:** for any  $\varepsilon > 0$  we can fix  $\eta > 0$  such that  $(\alpha(b) - \alpha(a))\eta < \varepsilon$ .  $f$  is uniformly continuous in  $[a, b]$  so

$$\forall \eta > 0 \exists \delta > 0 \text{ s.t. } \forall x, t \in [a, b] \ |x - t| < \delta \Rightarrow |f(x) - f(t)| < \eta$$

Now take a partition  $P$  of  $[a, b]$  such that  $\Delta x_i < \delta$  for any  $i$ , given that  $f$  is uniformly continuous this implies that  $M_i - m_i < \eta$  for any  $i$ . Then

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^n (M_i - m_i) \Delta \alpha_i \leq \eta \sum_{i=1}^n \Delta \alpha_i = \eta(\alpha(b) - \alpha(a)) < \varepsilon$$

So by the condition of existence  $f \in \mathcal{R}(\alpha)$ .

Q.E.D.

**Theorem 5 (Necessary Condition)** *If  $f$  is monotonic in  $[a, b]$  and  $\alpha$  is continuous then  $f \in \mathcal{R}(\alpha)$  in  $[a, b]$ .*

**Proof:** since  $\alpha$  is continuous, for any  $\varepsilon > 0$  choose a partition of  $[a, b]$  such that  $\Delta \alpha_i = \frac{\alpha(b) - \alpha(a)}{n}$  for any  $i$  and for some  $n \in \mathbb{N}$ . Then

$$U(P, f, \alpha) - L(P, f, \alpha) = \sum_{i=1}^n (f(x_i) - f(x_{i-1})) \Delta \alpha_i = \frac{\alpha(b) - \alpha(a)}{n} (f(b) - f(a)) < \varepsilon$$

it's enough to take  $n$  large enough. Alternatively we can say that since  $\alpha$  is uniformly continuous we can refine the partition until the condition is satisfied i.e.

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } \forall x_i \in P \ \Delta x_i < \delta \Rightarrow \Delta \alpha_i < \varepsilon$$

So by the condition of existence  $f \in \mathcal{R}(\alpha)$ .

Q.E.D.

Summing up:

1. bounded and continuous function  $f$  can be integrated with respect to any monotonic increasing function  $\alpha$ ;

2. bounded and monotonic function  $f$  can be integrated with respect to any monotonic increasing and continuous function  $\alpha$ .

**Theorem 6** *If  $f$  is discontinuous in  $E \subset [a, b]$  which is a set of a finite number of points, and  $\alpha$  is continuous in  $E$ , then  $f \in \mathcal{R}(\alpha)$  in  $[a, b]$ .*

**Theorem 7 (Properties)** *Assume  $f, f_1, f_2 \in \mathcal{R}(\alpha)$ , fix  $\alpha$  then*

$$1) \int_a^b (f_1 + f_2) d\alpha = \int_a^b f_1 d\alpha + \int_a^b f_2 d\alpha$$

$$2) \int_a^b c f d\alpha = c \int_a^b f d\alpha \quad \forall c \in \mathbb{R}$$

$$3) f_1 \leq f_2 \Rightarrow \int_a^b f_1 d\alpha \leq \int_a^b f_2 d\alpha$$

$$4) \forall c \in [a, b] \int_a^b f d\alpha = \int_a^c f d\alpha + \int_c^b f d\alpha$$

$$5) \left| \int_a^b f d\alpha \right| \leq \int_a^b |f| d\alpha$$

$$6) \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2 = \int_a^b f d(\alpha_1 + \alpha_2)$$

$$7) \int_a^b f d\alpha = - \int_b^a f d\alpha$$

Define a step function as

$$\mathbb{I}(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Assume that  $\alpha = \mathbb{I}(x - s)$  for some  $s \in (a, b)$ , and assume that  $f$  is continuous in  $s$ , then

$$\int_a^b f d\alpha = f(s)$$

Assume that  $\alpha = \sum_{n=1}^{\infty} c_n \mathbb{I}(x - s_n)$  for  $\{s_n\} \subset (a, b)$  and assume that  $\sum c_n$  converges, and  $f$  is continuous in  $[a, b]$ , then

$$\int_a^b f d\alpha = \sum_{n=0}^{\infty} c_n f(s_n)$$

**Theorem 8** *If  $\alpha$  is such that its derivative function  $\alpha' \in \mathcal{R}(\alpha)$ , then  $f \in \mathcal{R}(\alpha)$  if and only if  $f\alpha' \in \mathcal{R}(\alpha)$  and*

$$\int_a^b f d\alpha = \int_a^b f\alpha' dx$$

where the last is the Riemann integral.

Summing up:

1. if  $\alpha$  is a step function then the integral is a series;
2. if  $\alpha$  has an integrable derivative the integral is a simple Riemann integral (we can always reduce to this situation by taking  $\alpha(x) = x$  for which  $\alpha'(x) = 1$  and thus is integrable).

**Theorem 9 (Change of Variable)** *Consider a strictly increasing continuous function  $\varphi : [A, B] \rightarrow [a, b]$  such that  $\exists \varphi'$  and therefore  $\varphi'(x) > 0$ . Assume  $f \in \mathcal{R}(\alpha)$  in  $[a, b]$  and for a given function  $\alpha$  define*

$$\beta(y) = \alpha(\varphi(y)) \quad g(y) = f(\varphi(y))$$

then  $g \in \mathcal{R}(\beta)$  in  $[A, B]$  and

$$\int_a^b f d\alpha = \int_A^B g d\beta$$

Consider the special case  $\alpha(x) = x$  then  $\beta = \varphi$  then  $\varphi'$  is continuous and thus  $\varphi' \in \mathcal{R}$  in  $[A, B]$ .

Therefore

$$\int_a^b f(x) dx = \int_A^B f(\varphi(y))\varphi'(y) dy$$

We are changing the variable of integration by making the substitution  $x = \varphi(y)$  and noticing that

$$\frac{dx}{dy} = \frac{d\varphi(y)}{dy} = \varphi'(y)$$

and that  $a = \varphi(A)$  and  $b = \varphi(B)$ .

## The Fundamental Theorem of Calculus

**Theorem 10** Assume  $f \in \mathcal{R}$  in  $[a, b]$  then for any  $x \in [a, b]$  define the function:

$$F(x) = \int_a^x f(t) dt$$

We call  $F$  a primitive of  $f$  and it has the properties

1.  $F$  is continuous in  $[a, b]$ ;
2. if  $f$  is continuous in  $x_0 \in [a, b]$  then  $F$  is differentiable at  $x_0$  and  $F'(x_0) = f(x_0)$ , where
 
$$F'(x) \equiv \frac{dF(x)}{dx}.$$

**Proof:**  $f$  is bounded therefore  $|f(t)| \leq M$  for any  $t \in [a, b]$  moreover  $\forall \varepsilon > 0$  we can define  $\delta = \varepsilon/M$  and if we consider two points  $x, y \in [a, b]$  such that  $|y - x| < \delta$  then

$$|F(y) - F(x)| = \left| \int_x^y f(t) dt \right| = \sup_P \sum_{i=1}^n M_i \Delta x_i \leq M(y - x) < \varepsilon$$

So  $F$  is continuous in  $[a, b]$ .

If  $f$  is continuous in  $x_0$

$$\forall \varepsilon > 0 \exists \delta > 0 \text{ s.t. } \forall t \in [a, b] \quad |t - x_0| < \delta \Rightarrow |f(t) - f(x_0)| < \varepsilon$$

Now, take  $s \in [a, b]$  such that  $x_0 - \delta < s \leq x_0 \leq t < x_0 + \delta$  then we have

$$\left| \frac{F(t) - F(s)}{t - s} - f(x_0) \right| = \left| \frac{1}{t - s} \int_s^t [f(u) - f(x_0)] du \right| < \frac{1}{t - s} \int_s^t \varepsilon du = \varepsilon$$

Just take the limit for  $t \rightarrow s$  in the previous expression and we get  $F'(x_0) = f(x_0)$ .

Q.E.D.

**Theorem 11** *If  $f \in \mathcal{R}$  in  $[a, b]$  and  $F' = f$  then*

$$\int_a^b f(t) dt = F(b) - F(a)$$

Moreover

$$\int_a^x f(t) dt = F(x) - F(a)$$

and by differentiating both sides with respect to  $x$  we have  $f(x) = F'(x)$ .

Notice that if the extremes are more complex functions of  $x$  then we have

$$\int_{a(x)}^{b(x)} f(t) dt = F(b(x)) - F(a(x))$$

and the derivative of the previous expression is given by

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(t) dt = f(b(x))b'(x) - f(a(x))a'(x)$$

We can also introduce the notion of indefinite integral where we do not specify the extremes of the integration. So if  $f \in \mathcal{R}$  in  $[a, b]$  and  $F' = f$  then

$$\int f dx = F + c$$

The indefinite integral is always defined up to a constant  $c$ .

**Theorem 12 (Integration by Parts)** *Assume to have  $f \in \mathcal{R}$  in  $[a, b]$  and  $g \in \mathcal{R}$  in  $[a, b]$*

such that  $F' = f$  and  $G' = g$  then

$$\int_a^b F(t)g(t)dt = F(b)G(b) - F(a)G(a) - \int_a^b f(t)G(t)dt$$

**Proof:** to prove just define  $H(x) = F(x)G(x)$  then

$$F(b)G(b) - F(a)G(a) = H(b) - H(a) = \int_a^b H'(t)dt = \int_a^b (F'(t)G(t) + F(t)G'(t))dt$$

Q.E.D.

For the indefinite integral the integration by parts reads as:

$$\int FG'dx = FG - \int F'Gdx$$

## Improper Integrals

What if the function to be integrated is not bounded or  $[a, b]$  is not bounded?

We can try to define the integral as the limit of a Riemann integral. Notice however that the limits may not exist and in that case we say that the integral diverges.

1. If  $f$  diverges in a point  $c \in [a, b]$  we can write

$$\int_a^b f(t)dt = \lim_{\omega \rightarrow 0^-} \int_a^{c-\omega} f(t)dt + \lim_{\omega \rightarrow 0^+} \int_{c+\omega}^b f(t)dt$$

2. If instead either  $a = -\infty$  or  $b = +\infty$  or both we can write for some  $c \in \mathbb{R} / \{-\infty, +\infty\}$

$$\int_{-\infty}^{+\infty} f(t)dt = \lim_{k \rightarrow -\infty} \int_k^c f(t)dt + \lim_{k \rightarrow +\infty} \int_c^k f(t)dt$$

**Examples** Take  $f(x) = \frac{1}{\sqrt{x}}$  then its integral is not defined in  $x = 0$  however we can extend the Riemann integral as

$$\int_0^a \frac{1}{\sqrt{t}} dt = \lim_{\omega \rightarrow 0^+} \int_{\omega}^a \frac{1}{\sqrt{t}} dt = \lim_{\omega \rightarrow 0^+} 2(\sqrt{1} - \sqrt{\omega}) = 2$$

Now take  $f(x) = \frac{1}{x^2}$  integrate it from 1 to  $+\infty$ , we can extend the Riemann integral as

$$\int_1^{+\infty} \frac{1}{t^2} dt = \lim_{\omega \rightarrow +\infty} \int_1^{\omega} \frac{1}{t^2} dt = \lim_{\omega \rightarrow +\infty} \left( -\frac{1}{\omega} + 1 \right) = 1$$

In general we have the following conditions for defining an improper integral

$$\int_0^b \frac{1}{t^p} dt = \lim_{\omega \rightarrow 0^+} \int_{\omega}^b \frac{1}{t^p} dt \quad \Leftrightarrow \quad p < 1$$

$$\int_a^{+\infty} \frac{1}{t^p} dt = \lim_{\omega \rightarrow +\infty} \int_a^{\omega} \frac{1}{t^p} dt \quad \Leftrightarrow \quad p > 1$$

## Main Reference

Rudin, W. *Principles of Mathematical Analysis* McGraw-Hill, Inc. 1976