Basic definitions and theorems of topology

Definition 1 (Metric Space and Distance) *A set* X *is a metric space if* $\forall\, p, q \in X$ *we can define a real number* $d(p, q)$ *called distance (or metric) such that*

1.	$d(p, q) > 0$ *if* $p \neq q$ $\;d(p, p) = 0$	*positive definite*
2.	$d(p, q) = d(q, p)$	*symmetric*
3.	$d(p, q) < d(p, r) + d(q, r)\ \forall\, r \in X$	*triangular inequality*

The Euclidean spaces $\mathbb{R}^n$ are metric spaces where for a couple of points $\mathbf{x} = (x_1 \ldots x_n)$ and $\mathbf{y} = (y_1 \ldots y_n)$ the distance is defined as

$$d(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}|| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

Definition 2 (Ball in $\mathbb{R}^n$) *If* $\mathbf{x} \in \mathbb{R}^n$ *the open(closed) ball with center in* $\mathbf{x}$ *and radius* $\mathbf{r}$ *is defined as the set*

$$B(\mathbf{x}, \mathbf{r}) = \{\mathbf{y} \in \mathbb{R}^n \,:\, ||\mathbf{y} - \mathbf{x}|| < \mathbf{r} \ (or\ ||\mathbf{y} - \mathbf{x}|| \leq \mathbf{r})\}$$

Definition 3 (Convex Set) *A set* $E \subset X$ *is convex if* $\forall\, \mathbf{x}, \mathbf{y} \in E$ *and* $\lambda \in (0, 1)$*:*
$\lambda\mathbf{x} + (1 - \lambda\mathbf{y}) \in E$

Definition 4 (Neighborhood) *In a metric space* X*, a neighborhood of* $p \in X$ *is the set* *defined as*

$$U(p,r) = \{q \in X \, : \, d(p,q) < r \; for \; some \; r > 0\}$$

In $\mathbb{R}^n$*, the open ball* $B(\mathbf{x}, \mathbf{r})$ *is the neighborhood of* $\mathbf{x}$*.*

Definition 5 (Definitions of Points) *In a metric space* X

1. $p \in E \subset X$ *is a limit point of* E *if every of its neighborhoods* $U(p,r)$ *contains a* $q \neq p$ *such that* $q \in E$ *(it should hold for any* $r > 0$*);*[1]
2. $p \in E \subset X$ *is an interior point of* E *if there is a neighborhood* $U(p,r)$ *of* p *such that* $U \subset E$ *(it should exist at least one* $r > 0$*).*[2]

Definition 6 (Closed Set) *In a metric space* X*,* $E \subset X$ *is closed if every limit point of* E *is a point of* E*.*

Definition 7 (Open Set) *In a metric space* X*,* $E \subset X$ *is open if every point of* E *is an interior point of* E*.*

Definition 8 (Complementary Set) *In a metric space* X*, the complement of* $E \subset X$*, denoted as* E^c*, is the set of all points* $p \in X$ *such that* $p \notin X$

Theorem 1 *In a metric space* X*, every neighborhood is an open set.*

Theorem 2 *In a metric space* X*, a set* $E \subset X$ *is open if and only if* E^c *is closed. Therefore a set* $F \subset X$ *is closed if and only if* F^c *is open.*

Theorem 3 *In a metric space* X*, for any collection* $\{G_\alpha\}$ *of open subsets of* X*, and* $\{F_\alpha\}$ *of*

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[1] Intuitively given p we can find a sequence $\{q_n\}_{n=1}^{\infty}$ close enough to p for n large enough.

[2] Intuitively p is not on the border of E the closer p is to the border the smaller will be the radius of the neighborood but it will always be positive.

closed subsets of X, we have that

1. $\cup_\alpha G_\alpha$ *is open*
2. $\cap_\alpha F_\alpha$ *is closed*
3. $\cap_{i=1}^n G_i$ *is open*
4. $\cup_{i=1}^n F_i$ *is closed*

Definition 9 (Closure) *In a metric space X, if $E \subset X$ and we define E' as the set of all limit points of E, the closure of E denoted as $\bar{E}$ is defined as $\bar{E} = E \cup E'$.*

Theorem 4 *In a metric space X, given $E \subset X$, then*

1. *$\bar{E}$ is closed;*
2. *$E = \bar{E}$ if and only if E is closed;*
3. *$\bar{E} \subset F$ for every closed set $F \subset X$ such that $E \subset F$.*

Definition 10 (Open Cover) *In a metric space X, the open cover of $E \subset X$ is defined as a collection $\{G_\alpha\}$ of open subsets of X such that $E \subset \cup_\alpha G_\alpha$.*

Definition 11 (Compact Set) *In a metric space X, $K \subset X$ is compact if every open cover $\{G_\alpha\}$ of K, contains a finite subcover, i.e. there exist a finite set of indexes $(\alpha_1 \dots \alpha_n)$ such that $K \subset \cup_{i=1}^n G_{\alpha_i}$.*

Definition 12 (Bounded Set) *In a metric space X, $E \subset X$ is bounded if $\exists\, M \in \mathbb{R}$ and $\exists\, q \in X$ such that $d(p,q) < M \ \forall\, p \in E$.*

Theorem 5 *Compact subsets of metric spaces are closed.*

Theorem 6 *Closed subsets of compact sets are compact.*

Theorem 7 (Heine-Borel) *For a set $E \subset \mathbb{R}^n$ the following are equivalent*

1. *E is closed and bounded;*
2. *E is compact;*
3. *every infinite subset of E has a limit point in E.*

Theorem 8 (Weierstrass) *Every bounded infinite subset of $\mathbb{R}^n$ has a limit point in $\mathbb{R}^n$.*

Definition 13 (Connected Set) *In a metric space X, two subsets A and B are separated if both $A \cap \bar{B} = \emptyset$ and $\bar{A} \cap B = \emptyset$. A set $E \subset X$ is connected if E is not the union of two nonempty separated sets.*

Theorem 9 *A set $E \subset \mathbb{R}$ is connected if and only if $\forall\, x, y \in E$ then $x < z < y$ implies $z \in E$.*

Reference

Rudin, W. *Principles of Mathematical Analysis* McGraw-Hill, Inc. 1976