Basic definitions and theorems of numerical sequences

Definition 1 (Sequence) *A sequence in a metric space X is a function defined as*

$$\{a_n\} : \mathbb{N} \to X$$

$X \equiv \mathbb{R}$ *but it could be also* $\mathbb{R}^k$ *or* $\mathbb{C}$.

Definition 2 (Convergence) *A sequence $\{a_n\}$ in a metric space X converges if*

$$\exists\, a \in X \ s.t.\ \forall\, \epsilon > 0\ \exists\, N \in \mathbb{N}\ s.t.\ n \geq N \Rightarrow d(a_n, a) < \epsilon$$

In this case we write

$$\lim_{n\to\infty} a_n = a$$

Theorem 1 *If $a \in X$ and $a' \in X$ such that $\lim a_n = a$ and $\lim a_n = a'$ then $a = a'$.*

Theorem 2 *Suppose $\{a_n\}$ and $\{b_n\}$ are sequences in X and $\lim a_n = a$ and $\lim b_n = b$. Then*

1. $\lim a_n + b_n = a + b$;
2. $\lim ca_n = ca$, *and* $\lim c + a_n = c + a$, $\forall c \in \mathbb{R}$;
3. $\lim a_n b_n = ab$;
4. $\lim \frac{1}{a_n} = \frac{1}{a}$ *if* $a_n \neq 0\ \forall n$ *and* $a \neq 0$.

Definition 3 (Subsequence) *Given a sequence* $\{a_n\}$ *consider an increasing sequence* $\{n_k\}$ *of positive integers. Then* $\{a_{n_k}\}$ *is called a subsequence of* $\{a_n\}$*. Therefore* $\{a_n\}$ *converges to* a *if and only if any subsequence* $\{a_{n_k}\}$ *converges to* a*.*

Theorem 3
1. *If* $\{a_n\}$ *is a sequence in a compact metric space* X*, then some subsequence of* $\{a_n\}$ *converges to a point of* X*;*
2. *every bounded sequence in* $\mathbb{R}^k$ *contains a convergent subsequence.*

Definition 4 (Cauchy Sequence) *A sequence* $\{a_n\}$ *in a metric space* X *is said to be a Cauchy sequence if*

$$\forall \epsilon > 0\, \exists N \in \mathbb{N}\ s.t.\ n \geq N\ and\ m \geq N \Rightarrow\ d(a_n, a_m) < \epsilon$$

Theorem 4
1. *In any metric space* X *every convergent sequence is a Cauchy sequence;*
2. *if* X *is a compact metric space and if* $\{a_n\}$ *is a Cauchy sequence in* X*, then* $\{a_n\}$ *converges to some point of* X*;*
3. *in* $\mathbb{R}^k$ *every Cauchy sequence converges.*

Definition 5 (Complete Space) *Every metrc space in which every Cauchy sequence converges is said to be complete.*

All compact metric spaces and all Euclidean spaces are complete.
Every closed subset of a complete metric space is complete.

Definition 6 *A sequence* $\{a_n\} \subset \mathbb{R}$ *is said to be*

1. *monotonically incresing if* $a_n \leq a_{n+1}$, $\forall n$*;*
2. *monotonically decreasing if* $a_n \geq a_{n+1}$, $\forall n$*.*

Theorem 5 *Suppose* $\{a_n\}$ *is monotonic. Then* $\{a_n\}$ *converges if and only if is bounded.*

Theorem 6 *1. If* $p > 0$*, then* $\lim \frac{1}{n^p} = 0$*;*

2. if $p > 0$*, then* $\lim \sqrt[n]{p} = 1$*;*

3. $\lim \sqrt[n]{n} = 1$*;*

4. if $p > 0$ *and* $\alpha \in \mathbb{R}$*, then* $\lim \frac{n^\alpha}{(1+p)^n} = 0$*;*

5. if $|x| < 1$*, then* $\lim x^n = 0$*.*

Reference

Rudin, W. *Principles of Mathematical Analysis* McGraw-Hill, Inc. 1976