Basic definitions on Euclidean Spaces

Definition 1 (Euclidean Space) *An n dimensional Euclidean space is defined as*

$$\mathbb{R}^n = \{(x_1 \dots x_n) \ s.t. \ x_i \in \mathbb{R} \ \forall i\}$$

If $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$ and $a \in \mathbb{R}$ then

1. $\mathbf{x} + \mathbf{y} \in \mathbb{R}^n$;

2. $a\mathbf{x} \in \mathbb{R}^n$.

Therefore $\mathbb{R}^n$ is a vector space and $\mathbf{x}$ is a vector.

Definition 2 (Euclidean Distance) *The distance is a function d defined on $\mathbb{R}^n \times \mathbb{R}^n$ such that for any $\mathbf{x}$ and $\mathbf{y}$*

$$d(\mathbf{x}, \mathbf{y}) \equiv ||\mathbf{x} - \mathbf{y}|| = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

When $\mathbf{y} = \mathbf{0}$ then we have the norm of vector $\mathbf{x}$

$$||\mathbf{x}|| = \sqrt{\sum_{i=1}^{n} x_i^2}$$

Definition 3 (Ball) *For any $\mathbf{p} \in \mathbf{R}^n$ and $r \in \mathbb{R}^+$ we define the a ball with center in | and radius r as the open set*

$$B^n(\mathbf{p}, r) = \{\mathbf{x} \in \mathbb{R}^n \ s.t. \ ||\mathbf{x} - \mathbf{p}|| < r\}$$

Definition 4 (Basis) *Every vector* $\mathbf{x} \in \mathbb{R}^n$ *can be written as*

$$\mathbf{x} = \sum_{i=1}^{n} x_i \mathbf{e}_i$$

where $x_i \in \mathbb{R}$ *for any* i *and* $\mathbf{e}_i = (0 \ldots \underbrace{1}_{i} \ldots 0) \in \mathbb{R}^n$.
The set $\{\mathbf{e}_i\}$ *is called standard basis and it identifies the orthogonal axis. It is an indpendent set, i.e.* $\sum_i c_i \mathbf{e}_i = 0$ *if and only if* $c_i = 0$ *for any* i. *The standard basis is unique.*

Since $\mathbf{x}$ is a vector it will have

1. a magnitude (length) which is $||\mathbf{x}||$;
2. an orientation with respect to the axis (i.e. the standard basis) defined as $\cos \alpha_i = \frac{x_i}{||\mathbf{x}||}$

Definition 5 (Scalar Product) *Given* $\mathbf{x},\ \mathbf{y} \in \mathbb{R}^n$ *the scalar product is the real number defined as*

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i$$

It has the following properties

1. $\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$;
2. $\langle \mathbf{x}, \mathbf{y} + \mathbf{z} \rangle = \langle \mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{x}, \mathbf{z} \rangle$;
3. $\alpha \langle \mathbf{x}, \mathbf{y} \rangle = \alpha \langle \mathbf{x}, \mathbf{y} \rangle$;
4. $\langle \mathbf{0}, \mathbf{x} \rangle = 0$;
5. $\langle \mathbf{x}, \mathbf{x} \rangle = ||\mathbf{x}|| \geq 0$, *and* $\langle \mathbf{x}, \mathbf{x} \rangle = 0$ *if and only if* $\mathbf{x} = \mathbf{0}$.

The angle between two vectors can then be written as

$$\cos \theta = \frac{\langle \mathbf{x}, \mathbf{y} \rangle}{||\mathbf{x}|| ||\mathbf{y}||}$$

Theorem 1 (Cauchy-Schwarz Inequality) *Given* $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq ||\mathbf{x}|| ||\mathbf{y}||$$

Definition 6 (Orthogonality) *Given* $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$ *we say that they are orthogonal* $\mathbf{x} \perp \mathbf{y}$ *if and only if* $\langle \mathbf{x}, \mathbf{y} \rangle = 0$. *Notice that the vectors of the standard basis are orthogonal, i.e.* $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = 0$ *for* $i \neq j$, *while* $\langle \mathbf{e}_i, \mathbf{e}_i \rangle = ||\mathbf{e}_i|| = 1$. *The standard basis is made of orthonormal vectors.*

Theorem 2 (Triangular Inequality) *Given* $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$

$$||\mathbf{x} + \mathbf{y}|| \leq ||\mathbf{x}|| + ||\mathbf{y}||$$

Then if $\mathbf{x} \perp \mathbf{y}$ *we have the Pythagora's theorem*

$$||\mathbf{x} + \mathbf{y}||^2 = \langle \mathbf{x} + \mathbf{y}, \mathbf{x}\mathbf{y} \rangle = ||\mathbf{x}||^2 + ||\mathbf{y}||^2 + \underbrace{2\langle \mathbf{x}, \mathbf{y} \rangle}_{0}$$

Definition 7 (Orthogonal Projection) *Given* $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$ *and* $\mathbf{y} \neq \mathbf{0}$ *we define the normalized vector* $\mathbf{u} = \frac{\mathbf{y}}{||\mathbf{y}||}$. *The orthogonal projection of* $\mathbf{x}$ *onto* $\mathbf{y}$ *is*

$$\mathbf{w} = (\langle \mathbf{x}, \mathbf{u} \rangle)\mathbf{u}$$

In the special case $\mathbf{y} = \mathbf{e}_k$ *for some* k, *we have* $\mathbf{w} = x_k \mathbf{e}_k$. *Moreover* $||\mathbf{w}|| = \langle \mathbf{x}, \mathbf{u} \rangle$.

Linear Transformations

Definition 8 *A linear transformation* A *is defined as*

$$A : \mathbb{R}^n \rightarrow \mathbb{R}^m$$

such that for any $\mathbf{x}_1$ *and* $\mathbf{x}_2 \in \mathbb{R}^n$ *and for any* $c \in \mathbb{R}$ *we have*

1. $A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2$;

2. $A(c\mathbf{x}) = cA(\mathbf{x})$.

The action of A is determined by its action on a basis. Take $\{\mathbf{e}_i\}$ then if $\mathbf{x} = sum_i c_i \mathbf{e}_i$ we have

$$A(\mathbf{x}) = \sum_{i=1}^{n} c_i A(\mathbf{e}_i)$$

If $n = m$ and A is bijective then A is invertible.

Definition 9 *Take a basis* $\{\mathbf{e}_i\} \subset \mathbb{R}^n$ *and a basis* $\{\mathbf{u}_i\} \subset \mathbb{R}^m$ *then*

$$A(\mathbf{e}_j) = \sum_{i=1}^{m} a_{ij} \mathbf{u}_i \quad j = 1, \ldots, n$$

The matrix representation of A *is simply given by*

$$\begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn} \end{bmatrix} = (A\mathbf{e}_1 \ldots A\mathbf{e}_n)$$

Therefore $A(\mathbf{x}) \equiv A\mathbf{x}$ *and*

$$A\mathbf{x} = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} c_j \right) \mathbf{u}_i$$

The linear trasformations can be written as matrices, however notice that the matrix representation depends on the chosen basis.

Reference

Rudin, W. *Principles of Mathematical Analysis* McGraw-Hill, Inc. 1976